Berkeley Math Tournament , 2020 Fall, Algebra, Geometry, Individual, Team, Discrete Round

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–	Algebra Round
1	Marisela is putting on a juggling show! She starts with 1 ball, tossing it once per second. Lawrence tosses her another ball every five seconds, and she always tosses each ball that she has once per second. Compute the total number of tosses Marisela has made one minute after she starts juggling.
2	Let a and b be the roots of the polynomial $x^2 + 2020x + c$. Given that $\frac{a}{b} + \frac{b}{a} = 98$, compute $\sqrt{c}$.
3	The graph of the degree 2021 polynomial $P(x)$, which has real coefficients and leading coefficient 1, meets the x-axis at the points $(1, 0)$, $(2, 0)$, $(3, 0)$, $\ldots$, $(2020, 0)$ and nowhere else. The mean of all possible values of $P(2021)$ can be written in the form $a!/b$, where a and b are positive integers and a is as small as possible. Compute $a + b$.
4	Let φ be the positive solution to the equation $$x^2 = x + 1.$$ For $n \geq 0$, let a_n be the unique integer such that $\varphi^n - a_n\varphi$ is also an integer. Compute $$\sum_{n=0}^{10} a_n.$$
5	Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}+,\ f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$, where $\mathbb{R}^+$ represents the positive real numbers. Given that $f(2) = 3$, compute the last two digits of $f\left(2^{2^{2020}}\right)$.
6	Given that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the value of $$\sum_{n=3}^{10} \frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}}$$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

7 Let a, b, and c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=5$. The value of

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

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8 Compute the smallest value C such that the inequality

$$x^2(1+y)+y^2(1+x)\leq\sqrt{(x^4+4)(y^4+4)}+C$$

holds for all real x and y.

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9 There is a unique triple (a,b,c) of two-digit positive integers a, b, and c that satisfy the equation

$$a^3+3b^3+9c^3=9abc+1.$$

Compute $a+b+c$.

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10 For $k\geq 1$, define $a_k=2^k$. Let

$$S=\sum_{k=1}^{\infty}\cos^{-1}\left(\frac{2a_k^2-6a_k+5}{\sqrt{(a_k^2-4a_k+5)(4a_k^2-8a_k+5)}}\right).$$

Compute $\lfloor 100S\rfloor$.

Tie 1 Find the sum of the squares of all values of x that satisfy $\log_2(x+3)+\log_2(2-x)=2$.

Tie 2 The polynomial $f(x)=x^3+rx^2+sx+t$ has r, s, and t as its roots (with multiplicity), where $f(1)$ is rational and $t\neq 0$. Compute $|f(0)|$.

Tie 3 Let x and y be integers from -10 to 10, inclusive, with $xy\neq 1$. Compute the number of ordered pairs (x,y) such that

$$\left|\frac{x+y}{1-xy}\right|\leq 1.$$

– Geometry Round

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1 A Yule log is shaped like a right cylinder with height 10 and diameter 5. Freya cuts it parallel to its bases into 9 right cylindrical slices. After Freya cut it, the combined surface area of the slices of the Yule log increased by $a\pi$. Compute a.

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2 Let O be a circle with diameter $AB = 2$. Circles O_1 and O_2 have centers on $\overline{AB}$ such that O is tangent to O_1 at A and to O_2 at B, and O_1 and O_2 are externally tangent to each other. The minimum possible value of the sum of the areas of O_1 and O_2 can be written in the form $\frac{m\pi}{n}$ where m and n are relatively prime positive integers. Compute $m + n$.

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3 Right triangular prism $ABCDEF$ with triangular faces $\triangle ABC$ and $\triangle DEF$ and edges $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ has $\angle ABC = 90^o$ and $\angle EAB = \angle CAB = 60^o$. Given that $AE = 2$, the volume of $ABCDEF$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
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`png`

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4 Alice is standing on the circumference of a large circular room of radius 10. There is a circular pillar in the center of the room of radius 5 that blocks Alice's view. The total area in the room Alice can see can be expressed in the form $\frac{m\pi}{n} + p\sqrt{q}$, where m and n are relatively prime positive integers and p and q are integers such that q is square-free. Compute $m + n + p + q$. (Note that the pillar is not included in the total area of the room.)
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5 Let $A_1 = (0, 0)$, $B_1 = (1, 0)$, $C_1 = (1, 1)$, $D_1 = (0, 1)$. For all $i > 1$, we recursively define

$$A_i = \frac{1}{2020}(A_{i-1} + 2019B_{i-1}), B_i = \frac{1}{2020}(B_{i-1} + 2019C_{i-1})$$

$$C_i = \frac{1}{2020}(C_{i-1} + 2019D_{i-1}), D_i = \frac{1}{2020}(D_{i-1} + 2019A_{i-1})$$

where all operations are done coordinate-wise.
`https://cdn.artofproblemsolving.com/attachments/8/7/9b6161656ed2bc70510286dc8cb75cc5bde6c`
`png`
If $[A_iB_iC_iD_i]$ denotes the area of $A_iB_iC_iD_i$, there are positive integers a, b, and c such that $\sum_{i=1}^{\infty}[A_iB_iC_iD_i] = \frac{a^2b}{c}$, where b is square-free and c is as small as possible. Compute the value of $a + b + c$

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6 A tetrahedron has four congruent faces, each of which is a triangle with side lengths 6, 5, and 5. If the volume of the tetrahedron is V , compute V^2
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7 Circle Γ has radius 10, center O, and diameter $\overline{AB}$. Point C lies on Γ such that $AC = 12$. Let P be the circumcenter of $\triangle AOC$. Line AP intersects Γ at Q, where Q is different from A. Then the value of $\frac{AP}{AQ}$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

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8 Let triangle $\triangle ABC$ have $AB = 17$, $BC = 14$, $CA = 12$. Let M_A, M_B, M_C be midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively. Let the angle bisectors of A, B, and C intersect $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at P, Q, and R, respectively. Reflect M_A about $\overline{AP}$, M_B about $\overline{BQ}$, and M_C about $\overline{CR}$ to obtain M'_A, M'_B, M'_C, respectively. The lines AM'_A, BM'_B, and CM'_C will then intersect $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at D, E, and F, respectively. Given that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur at a point K inside the triangle, in simplest form, the ratio $[KAB] : [KBC] : [KCA]$ can be written in the form $p : q : r$, where p, q and r are relatively prime positive integers and $[XYZ]$ denotes the area of $\triangle XYZ$. Compute $p + q + r$.

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9 The Fibonacci numbers F_n are defined as $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Let A be the minimum area of a (possibly degenerate) convex polygon with 2020 sides, whose side lengths are the first 2020 Fibonacci numbers F_1, F_2, ... , F_{2020} (in any order). A degenerate convex polygon is a polygon where all angles are $\leq 180^o$. If A can be expressed in the form

$$\frac{\sqrt{(F_a - b)^2 - c}}{d}$$

, where a, b, c and d are positive integers, compute the minimal possible value of $a + b + c + d$.

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10 Let E be an ellipse where the length of the major axis is 26, the length of the minor axis is 24, and the foci are at points R and S. Let A and B be points on the ellipse such that $RASB$ forms a non-degenerate quadrilateral, lines RA and SB intersect at P with segment PR containing A, and lines RB and AS intersect at Q with segment QR containing B. Given that $RA = AS$, $AP = 26$, the perimeter of the non-degenerate quadrilateral $RPSQ$ is $m + \sqrt{n}$, where m and n are integers. Compute $m + n$.

Tie 1 Given a regular hexagon, a circle is drawn circumscribing it and another circle is drawn inscribing it. The ratio of the area of the larger circle to the area of the smaller circle can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Tie 2 Quadrilateral $ABCD$ is cyclic with $AB = CD = 6$. Given that $AC = BD = 8$ and $AD + 3 = BC$, the area of $ABCD$ can be written in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that p and r are relatively prime and that q is square-free. Compute $p + q + r$.

Tie 3 In unit cube $ABCDEFGH$ (with faces $ABCD$, $EFGH$ and connecting vertices labeled so that $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, $\overline{DH}$ are edges of the cube), L is the midpoint of GH. The area of $\triangle CAL$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

– Team Round

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1 Julia and James pick a random integer between 1 and 10, inclusive. The probability they pick the same number can be written in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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2 There are 38 people in the California Baseball League (CBL). The CBL cannot start playing games until people are split into teams of exactly 9 people (with each person in exactly one team). Moreover, there must be an even number of teams. What is the fewest number of people who must join the CBL such that the CBL can start playing games? The CBL may not revoke membership of the 38 people already in the CBL.

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3 An ant is at one corner of a unit cube. If the ant must travel on the box's surface, the shortest distance the ant must crawl to reach the opposite corner of the cube can be written in the form $\sqrt{a}$, where a is a positive integer. Compute a.

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4 Let $p(x) = 3x^2 + 1$. Compute the largest prime divisor of $p(100) - p(3)$

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5 Call a positive integer *prime-simple* if it can be expressed as the sum of the squares of two distinct prime numbers. How many positive integers less than or equal to 100 are prime-simple?

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6 Jack writes whole numbers starting from 1 and skips all numbers that contain either a 2 or 9. What is the 100th number that Jack writes down?

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7 A fair six-sided die is rolled five times. The probability that the five die rolls form an increasing sequence where each value is strictly larger than the one that preceded can be written in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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8 Let $ABCD$ be a unit square and let E and F be points inside $ABCD$ such that the line containing $\overline{EF}$ is parallel to $\overline{AB}$. Point E is closer to $\overline{AD}$ than point F is to $\overline{AD}$. The line containing $\overline{EF}$ also bisects the square into two rectangles of equal area. Suppose $[AEFB] = [DEFC] = 2[AED] = 2[BFC]$. The length of segment $\overline{EF}$ can be expressed as m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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9 A sequence a_n is defined by $a_0 = 0$, and for all $n \geq 1$, $a_n = a_{n-1} + (-1)^n \cdot n^2$. Compute a_{100}

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10 How many integers $100 \leq x \leq 999$ have the property that, among the six digits in $\lfloor 280 + \frac{x}{100} \rfloor$ and x, exactly two are identical?

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11 Compute $\sum_{x=1}^{999} \gcd(x, 10x + 9)$.

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12 A hollow box (with negligible thickness) shaped like a rectangular prism has a volume of 108 cubic units. The top of the box is removed, exposing the faces on the inside of the box. What is the minimum possible value for the sum of the areas of the faces on the outside and inside of the box?

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13 Compute the expected sum of elements in a subset of $\{1, 2, 3, ..., 2020\}$ (including the empty set) chosen uniformly at random.

14 In the star shaped figure below, if all side lengths are equal to 3 and the three largest angles of the figure are 210 degrees, its area can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and that b is square-free. Compute $a + b + c$.
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`png`

15 Consider a random string s of 10^{2020} base-ten digits (there can be leading zeroes). We say a substring s' (which has no leading zeroes) is self-locating if s' appears in s at index s' where the string is indexed at 1. For example the substring 11 in the string "122352242411" is selflocating since the 11th digit is 1 and the 12th digit is 1. Let the expected number of self-locating substrings in s be G. Compute $\lfloor G \rfloor$.

16 Let T be the answer to question 18. Rectangle $ZOMR$ has $ZO = 2T$ and $ZR = T$. Point B lies on segment ZO, O' lies on segment OM, and E lies on segment RM such that $BR = BE = EO'$, and $\angle BEO' = 90^o$. Compute $2(ZO + O'M + ER)$.

PS. You had better calculate it in terms of T.

17 Let T be the answer to question 16. Compute the number of distinct real roots of the polynomial $x^4 + 6x^3 + \frac{T}{2}x^2 + 6x + 1$.

18 Let T be the answer to question 17, and let $N = \frac{24}{T}$. Leanne flips a fair coin N times. Let X be the number of times that within a series of three consecutive flips, there were exactly two heads or two tails. What is the expected value of X?

19 John is flipping his favorite bottle, which currently contains 10 ounces of water. However, his bottle is broken from excessive flipping, so after he performs a flip, one ounce of water leaks out of his bottle. When his bottle contains k ounces of water, he has a $\frac{1}{k+1}$ probability of landing it on its bottom. What is the expected number of number of flips it takes for John's bottle to land on its bottom ?

20 Non-degenerate quadrilateral $ABCD$ with $AB = AD$ and $BC = CD$ has integer side lengths, and $\angle ABC = \angle BCD = \angle CDA$. If $AB = 3$ and $B \ne D$, how many possible lengths are there for BC?

21 Let $\triangle ABC$ be a right triangle with legs $AB = 6$ and $AC = 8$. Let I be the incenter of $\triangle ABC$ and X be the other intersection of AI with the circumcircle of $\triangle ABC$. Find $\overline{AI} \cdot \overline{IX}$.

22 Suppose that x, y, and z are positive real numbers satisfying

$$\begin{cases} x^2 + xy + y^2 = 64 \\ y^2 + yz + z^2 = 49 \\ z^2 + zx + x^2 = 57 \end{cases}$$

Then $\sqrt[3]{xyz}$ can be expressed as m/n , where m and n are relatively prime positive integers. Compute $m+n$.

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23 Let $0 < \theta < 2\pi$ be a real number for which $\cos(\theta)+\cos(2\theta)+\cos(3\theta)+...+\cos(2020\theta)=0$ and $\theta = \frac{\pi}{n}$ for some positive integer n. Compute the sum of the possible values of $n \le 2020$.

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24 For positive integers N and m, where $m \le N$, define

$$a_{m,N} = \frac{1}{\binom{N+1}{m}} \sum_{i=m-1}^{N-1} \frac{\binom{i}{m-1}}{N-i}$$

Compute the smallest positive integer N such that

$$\sum_{m=1}^{N} a_{m,N} > \frac{2020N}{N+1}$$

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25 Submit an integer between 1 and 50, inclusive. You will receive a score as follows:
If some number is submitted exactly once: If E is your number, A is the closest number to E which received exactly one submission, and M is the largest unique submission, you will receive $\frac{25}{M}(A-|E-A|)$ points, rounded to the nearest integer.
If no number was submitted exactly once: If E is your number, K is the number of people who submitted E, and M is the number of people who submitted the most commonly submitted number, then you will receive $\frac{25(M-K)}{M}$ points, rounded to the nearest integer.

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26 Estimate the value of the 2020th prime number p such that $p+2$ is also prime.

If $E > 0$ is your estimate and A is the correct answer, you will receive $25\min\left(\frac{E}{A}, \frac{A}{E}\right)^2$ points, rounded to the nearest integer. (An estimate less than or equal to 0 will receive 0 points.

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27 Estimate the number of 1s in the hexadecimal representation of $2020!$.

If E is your estimate and A is the correct answer, you will receive $\max(25-0.5|A-E|, 0)$ points, rounded to the nearest integer.

– Individual Round

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1 Justin throws a standard six-sided die three times in a row and notes the number of dots on the top face after each roll. How many different sequences of outcomes could he get?

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2 Let m be the answer to this question. What is the value of $2m-5$?

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3 At Zoom University, people's faces appear as circles on a rectangular screen. The radius of one's face is directly proportional to the square root of the area of the screen it is displayed on. Haydn's face has a radius of 2 on a computer screen with area 36. What is the radius of his face on a 16×9 computer screen?

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4 Let a, b, and c be integers that satisfy $2a + 3b = 52$, $3b + c = 41$, and $bc = 60$. Find $a + b + c$

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5 A Yule log is shaped like a right cylinder with height 10 and diameter 5. Freya cuts it parallel to its bases into 9 right cylindrical slices. After Freya cut it, the combined surface area of the slices of the Yule log increased by $a\pi$. Compute a.

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6 Haydn picks two different integers between 1 and 100, inclusive, uniformly at random. The probability that their product is divisible by 4 can be expressed in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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7 A square has coordinates at $(0, 0)$, $(4, 0)$, $(0, 4)$, and $(4, 4)$. Rohith is interested in circles of radius r centered at the point $(1, 2)$. There is a range of radii $a < r < b$ where Rohith's circle intersects the square at exactly 6 points, where a and b are positive real numbers. Then $b-a$ can be written in the form $m + \sqrt{n}$, where m and n are integers. Compute $m + n$.

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8 By default, iPhone passcodes consist of four base-10 digits. However, Freya decided to be unconventional and use hexadecimal (base-16) digits instead of base-10 digits! (Recall that $10_{16} = 16_{10}$.) She sets her passcode such that exactly two of the hexadecimal digits are prime. How many possible passcodes could she have set?

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9 A circle C with radius 3 has an equilateral triangle inscribed in it. Let D be a circle lying outside the equilateral triangle, tangent to C, and tangent to the equilateral triangle at the midpoint of one of its sides. The radius of D can be written in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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10 Given that p and $p^4 + 34$ are both prime numbers, compute p.

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11 Equilateral triangle ABC has side length 2. A semicircle is drawn with diameter BC such that it lies outside the triangle, and minor arc BC is drawn so that it is part of a circle centered at A. The area of the "lune" that is inside the semicircle but outside sector ABC can be expressed in the form $\sqrt{p} - \frac{q\pi}{r}$, where p, q, and r are positive integers such that q and r are relatively prime. Compute $p + q + r$.
`https://cdn.artofproblemsolving.com/attachments/7/7/f349a807583a83f93ba413bebf07e01326555`
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12 Compute the remainder when $98!$ is divided by 101.

13 Sheila is making a regular-hexagon-shaped sign with side length 1. Let $ABCDEF$ be the regular hexagon, and let R, S, T and U be the midpoints of FA, BC, CD and EF, respectively. Sheila splits the hexagon into four regions of equal width: trapezoids $ABSR$, $RSCF$, $FCTU$, and $UTDE$. She then paints the middle two regions gold. The fraction of the total hexagon that is gold can be written in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.
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=\&rn=MjAyMCBCTVQgSW5kaXZpZHVhbCAxMy5wbmc=

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14 Let B, M, and T be the three roots of the equation $x^3 + 20x^2 - 18x - 19 = 0$. What is the value of $|(B + 1)(M + 1)(T + 1)|$?

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15 The graph of the degree 2021 polynomial $P(x)$, which has real coefficients and leading coefficient 1, meets the x-axis at the points $(1, 0)$, $(2, 0)$, $(3, 0)$, ... , $(2020, 0)$ and nowhere else. The mean of all possible values of $P(2021)$ can be written in the form $a!/b$, where a and b are positive integers and a is as small as possible. Compute $a + b$.

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16 The triangle with side lengths $3, 5$, and k has area 6 for two distinct values of k: x and y. Compute $|x^2 - y^2|$.

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17 Shrek throws 5 balls into 5 empty bins, where each ball's target is chosen uniformly at random. After Shrek throws the balls, the probability that there is exactly one empty bin can be written in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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18 Let x and y be integers between 0 and 5, inclusive. For the system of modular congruences

$$\begin{cases} x + 3y \equiv 1 \ (mod\, 2) \\ 4x + 5y \equiv 2 \ (mod\, 3) \end{cases}$$

, find the sum of all distinct possible values of $x + y$

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19 Alice is standing on the circumference of a large circular room of radius 10. There is a circular pillar in the center of the room of radius 5 that blocks Alice's view. The total area in the room Alice can see can be expressed in the form $\frac{m\pi}{n} + p\sqrt{q}$, where m and n are relatively prime positive integers and p and q are integers such that q is square-free. Compute $m + n + p + q$. (Note that the pillar is not included in the total area of the room.)
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20 Compute the number of positive integers $n \leq 1890$ such that n leaves an odd remainder when divided by all of $2, 3, 5$, and 7.

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21 Let P be the probability that the product of 2020 real numbers chosen independently and uniformly at random from the interval $[-1, 2]$ is positive. The value of $2P - 1$ can be written in the form $\left(\frac{m}{n}\right)^b$, where m, n and b are positive integers such that m and n are relatively prime and b is as large as possible. Compute $m + n + b$.

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22 Three lights are placed horizontally on a line on the ceiling. All the lights are initially off. Every second, Neil picks one of the three lights uniformly at random to switch: if it is off, he switches it on; if it is on, he switches it off. When a light is switched, any lights directly to the left or right of that light also get turned on (if they were off) or off (if they were on). The expected number of lights that are on after Neil has flipped switches three times can be expressed in the form m/n , where m and n are relatively prime positive integers. Compute $m + n$.

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23 Circle Γ has radius 10, center O, and diameter AB. Point C lies on Γ such that $AC = 12$. Let P be the circumcenter of $\triangle\ AOC$. Line AP intersects Γ at Q, where Q is different from A. Then the value of $\frac{AP}{AQ}$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

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24 Let N be the number of non-empty subsets T of $S = \{1, 2, 3, 4, ..., 2020\}$ satisfying $\max(T) > 1000$. Compute the largest integer k such that 3^k divides N.

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25 Let $f : R^+ \to R^+$ be a function such that for all $x, y \in R^+$, $f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$, where R^+ represents the positive real numbers. Given that $f(2) = 3$, compute the last two digits of $f(2^{2^{2020}})$.
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Tie 1 An *exterior* angle is the supplementary angle to an interior angle in a polygon. What is the sum of the exterior angles of a triangle and dodecagon (12-gon), in degrees?

Tie 2 Let $\eta \in [0, 1]$ be a relative measure of material absorbence. η values for materials combined together are additive. η for a napkin is 10 times that of a sheet of paper, and a cardboard roll has $\eta = 0.75$. Justin can create a makeshift cup with $\eta = 1$ using 50 napkins and nothing else. How many sheets of paper would he need to add to a cardboard roll to create a makeshift cup with $\eta = 1$?

Tie 3 $\triangle\ ABC$ has $AB = 5$, $BC = 12$, and $AC = 13$. A circle is inscribed in $\triangle\ ABC$, and MN tangent to the circle is drawn such that M is on $\overline{AC}$, N is on $\overline{BC}$, and $\overline{MN} \parallel \overline{AB}$. The area of $\triangle\ MNC$ is m/n , where m and $n$are relatively prime positive integers. Find $m + n$.

Tie 4 In an 6×6 grid of lattice points, how many ways are there to choose 4 points that are vertices of a nondegenerate quadrilateral with at least one pair of opposite sides parallel to the sides of the grid?

Tie 5 The polynomial $f(x) = x^3 + rx^2 + sx + t$ has r, s, and t as its roots (with multiplicity), where $f(1)$ is rational and $t \neq 0$. Compute $|f(0)|$.

– Discrete Round

1 How many permutations of the set $\{B, M, T, 2, 0\}$ do not have B as their first element?

2 Haydn picks two different integers between 1 and 100, inclusive, uniformly at random. The probability that their product is divisible by 4 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

3 Compute the remainder when $98!$ is divided by 101.

4 Three lights are placed horizontally on a line on the ceiling. All the lights are initially off. Every second, Neil picks one of the three lights uniformly at random to switch: if it is off, he switches it on; if it is on, he switches it off. When a light is switched, any lights directly to the left or right of that light also get turned on (if they were off) or off (if they were on). The expected number of lights that are on after Neil has flipped switches three times can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

5 Let P be the probability that the product of 2020 real numbers chosen independently and uniformly at random from the interval $[-1, 2]$ is positive. The value of $2P - 1$ can be written in the form $\left(\frac{m}{n}\right)^b$, where m, n and b are positive integers such that m and n are relatively prime and b is as large as possible. Compute $m + n + b$.

6 Let N be the number of non-empty subsets T of $S = \{1, 2, 3, 4, ..., 2020\}$ satisfying $max(T) > 1000$. Compute the largest integer k such that 3^k divides N.

7 Compute the number of ordered triples of positive integers (a, b, c) such that $a + b + c + ab + bc + ac = abc + 1$.

8 Dexter is running a pyramid scheme. In Dexter's scheme, he hires ambassadors for his company, Lie Ultimate. Any ambassador for his company can recruit up to two more ambassadors (who are not already ambassadors), who can in turn recruit up to two more ambassadors each, and so on (Dexter is a special ambassador that can recruit as many ambassadors as he would like). An ambassador's downline consists of the people they recruited directly as well as the downlines of those people. An ambassador earns executive status if they recruit two new people and each of those people has at least 70 people in their downline (Dexter is not considered an executive). If there are 2020 ambassadors (including Dexter) at Lie Ultimate, what is the maximum number of ambassadors with executive status?

9 For any point (x, y) with $0 \leq x < 1$ and $0 \leq y < 1$, Jenny can perform a shuffle on that point, which takes the point to $(\{3x + y\}, \{x + 2y\})$ where $\{a\}$ denotes the fractional or decimal part of a (so for example,$\{\pi\} = \pi - 3 = 0.1415...$). How many points p are there such that after 3 shuffles on p, p ends up in its original position?

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10 Let $\psi(n)$ be the number of integers $0 \leq r < n$ such that there exists an integer x that satis es $x^2 + x \equiv r \pmod{n}$. Find the sum of all distinct prime factors of

$$\sum_{i=0}^{4}\sum_{j=0}^{4}\psi(3^i 5^j).$$

Tie 1 Compute the smallest positive integer n such that $\frac{n}{2}$ is a perfect square and $\frac{n}{3}$ is a perfect cube.

Tie 2 On a certain planet, the alien inhabitants are born without any arms, legs, or noses. Every year, on their birthday, each alien randomly grows either an arm, a leg, or a nose, with equal probability for each. After its sixth birthday, the probability that an alien will have at least 2 arms, at least 2 legs, and at least 1 nose on the day is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Tie 3 Three distinct integers a_1, a_2, a_3 between 1 and 21, inclusive, are selected uniformly at random. The probability that the greatest common factor of $a_i - a_j$ and 21 is 7 for some positive integers i and j, where $1 \leq i \neq j \leq 3$, can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
