Belarusian National Olympiad 2015

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by proximo

– Day 1

– Line intersects hyperbola H_1, given by the equation $y = 1/x$ at points A and B, and hyperbola H_2, given by the equation $y = -1/x$ at points C and D. Tangents to hyperbola H_1 at points A and B intersect at point M, and tangents to hyperbola H_2 at points C and D intersect at point N. Prove that points M and N are symmetric about the origin.

– A natural number n was alternately divided by 29, 41 and 59. The result was three nonzero remainders, the sum of which equals n. Find all such n

– Let A_1 be a midmoint of BC, and G is a centroid of the non-isosceles triangle $\triangle ABC$. $GBKL$ and $GCMN$ are the squares lying on the left with respect to rays GB and GC respectively. Let A_2 be a midpoint of a segment connecting the centers of the squares $GBKL$ and $GCMN$. Circumcircle of triangle $\triangle A_1A_2G$ intersects BC at points A_1 and X. Find $\frac{A_1X}{XH}$, where H is a base of altitude AH of the triangle $\triangle ABC$.

– Find all functions $f(x)$ determined on interval $[0,1]$, satisfying following conditions

$$\{f(x)\}\sin^2 x + \{x\}\cos f(x)\cos x = f(x)$$

$$f(f(x)) = f(x)$$

Here $\{y\}$ means a fractional part of number y

– Day 2

– Find all real $x \geq -1$ such that for all $a_1, ..., a_n \geq 1$, where $n \geq 2$ the following inequality holds

$$\frac{a_1 + x}{2} * \frac{a_2 + x}{2} * ... * \frac{a_n + x}{2} \leq \frac{a_1 a_2 ... a_n + x}{2}$$

– Let M be a set of natural numbers from 1 to 2015 which are not perfect squares.
a) Prove that for any $n \in M$ $\{\sqrt{n}\} \geq 0.011$
b) Prove that there exists number $n \in M$ such that $\{\sqrt{n}\} < 0.0115$
Here $\{y\}$ means the fractional part of number y

– Let I be an incenter of a triangle $\triangle ABC$. Points A_1, B_1, C_1 are the tangent points of the inscribed circle on sides BC, CA and AB respectively. Circumcircle of $\triangle BC_1B_1$ intersects line BC at points B and K and Circumcircle of $\triangle CB_1C_1$ intersects line BC at points C and L. Prove that lines LC_1, KB_1 and IA_1 are concurrent.

– Let n be a natural number. What is the least number m ($m > n$) such that the set of all natural numbers forn n to m (inclusively) can be divided into subsets such that in each subset one of the numbers equals the sum of other numbers in this subset?