Belarusian National Olympiad 2012

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by proximo

– Day 1

– Let AB and CD be two parallel chordes on hyperbola $y = 1/x$. Lines AC and BD intersect axis Oy at points A_1 and D_1 respectively, and axis Ox - at points C_1 and B_1 respectively. Prove that the area of $\triangle A_1OC_1$ equals the area of $\triangle D_1OB_1$

– For positive real nubers a, b, c find the maximum real number x, such that there exist positive numbers p, q, r, such that $p + q + r = 1$ and x does not exceed numbers $a\frac{p}{q}, b\frac{q}{r}, c\frac{r}{p}$

– Find all pairs (f, h) of functions $f, h : \mathbb{R} \to \mathbb{R}$ such as for all real x and y the equation holds.

$$f(x^2 + yh(x)) = xh(x) + f(xy)$$

– Inside the circle w of radius 1 there are n line segments with total length $2\sqrt{n}$. Prove that there exists a circle such that its center coincides with a center of w and it intersects at least two of line segments.

– Day 2

– Find all pairs $(n; p)$ of natural numbers n and prime numbers p, satisfying the equation

$$p(p - 1) = 2(n^3 + 1)$$

– Let point I be an incenter of $\triangle ABC$. Ray AI intersects circumcircle of $\triangle ABC$ at point D. Circumcircle of $\triangle CDI$ intersects ray BI at ponts I and K. Prove that $BK = CK$.

– a) In isosceles trapezoid from six line segments (four sides and two diagonals) three are colored red and three are colored green. Prove that from one of the triples of line segmets of the same colour it is possible to make a triangle.
b) Will the previous statement remain for arbitary trapezoid?

– $2n$ girls and $2n$ boys danced on the school ball. It's known, that for any pair of girls the amount of boys danced with only one of them equals n.
Prove that the previous statement is also true for boys.