www.artofproblemsolving.com/community/c2525986
by jasperE3, jbaca, blackbluecar

– Day 1

1 Let $P = \{p_1, p_2, \dots, p_{10}\}$ be a set of 10 different prime numbers and let A be the set of all the integers greater than 1 so that their prime decomposition only contains primes of P. The elements of A are colored in such a way that:

- each element of P has a different color,
- if $m, n \in A$, then mn is the same color of m or n,
- for any pair of different colors $\mathcal{R}$ and $\mathcal{S}$, there are no $j, k, m, n \in A$ (not necessarily distinct from one another), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, so that j is a divisor of m and n is a divisor of k, simultaneously.

Prove that there exists a prime of P so that all its multiples in A are the same color.

2 Consider an acute-angled triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of the triangle CEF and Γ meet at X and C, with $X \neq C$. The line BX and the tangent to Γ through A meet at Y. Let P be the point on segment AB so that $YP = YA$, with $P \neq A$, and let Q be the point where AB and the parallel to BC through Y meet each other. Show that F is the midpoint of PQ.

3 Let $a_1, a_2, a_3, \dots$ be a sequence of positive integers and let $b_1, b_2, b_3, \dots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n}, \text{ for } n \geq 1$$

Show that, if there exists at least one term among every million consecutive terms of the sequence $b_1, b_2, b_3, \dots$ that is an integer, then there exists some k such that $b_k > 2021^{2021}$.

– Day 2

4 Let a, b, c, x, y, z be real numbers such that

$$a^2 + x^2 = b^2 + y^2 = c^2 + z^2 = (a+b)^2 + (x+y)^2 = (b+c)^2 + (y+z)^2 = (c+a)^2 + (z+x)^2$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

5 For a finite set C of integer numbers, we define $S(C)$ as the sum of the elements of C. Find two non-empty sets A and B whose intersection is empty, whose union is the set $\{1, 2, \dots, 2021\}$ and such that the product $S(A)S(B)$ is a perfect square.

---

6 Consider a n-sided regular polygon, $n \geq 4$, and let V be a subset of r vertices of the polygon. Show that if $r(r-3) \geq n$, then there exist at least two congruent triangles whose vertices belong to V.
