Belarusian National Olympiad 2010

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by proximo

– Day 1

– Let M be the point of intersection of the diagonals AC and BD of trapezoid $ABCD$ ($BC||AD$), $AD > BC$. Circle w_1 passes through the point M and tangents AD at the point A. Circle w_2 passes through the point M and tangents AD at the point D. Point S is the point of intersection of lines AB and DC. Line AS intersects w_1 at the point X. Line DS intersects w_2 at the point Y. O is a center of a circumcircle of $\triangle ASD$. Prove that $SO \perp XY$

– Let r be a fixed positive real number. It is known that for some positive integer n the following statement is true: for any positive real numbers $a_1, ..., a_n$ satisfying the equation $a_1 + ... + a_n = r(\frac{1}{a_1} + ... + \frac{1}{a_n})$ they also satisfy the equation $\frac{1}{\sqrt{r}-a_1} + ... + \frac{1}{\sqrt{r}-a_n} = \frac{1}{\sqrt{r}}$ ($a_i \neq \sqrt{r}$). Find n.

– Nick and Mike are playing the following game. They have a heap and 330 stones in it. They take turns. In one turn it is allowed to take from the heap exactly 1, exactly n or exactly m stones. The boy who takes the last stone wins. Before the beginning Nick says the number n, ($1 < n < 10$). After that Mike says the number m, ($m \neq n$, $1 < m < 10$). Nick goes first. Is it possible for one of the boys to say the number to provide the victory to himself, regardless of his opponents number and strategy?

– There are 15 points on the plane, coloured blue, red and green. It is known that the sum of all pairwise distances between red and blue points equals 51, between red and green points equals 39, between blue and green points equals 1. What are the possible amounts of red, blue and green points?

– Day 2

– Let $\sigma(n)$ denote the sum and $\tau(n)$ denote the amount of natural divisors of number n (including 1 and n). Find the greatest real number a such that for all $n > 1$ the following inequality is true:

$$\frac{\sigma(n)}{\tau(n)} \geq a\sqrt{n}$$

– Let O_1 and O_2 be the centers of circles w_1, w_2 respectively. Circle w_1 intersects circle w_2 at points C and D. Line O_1O_2 intersects circle w_2 at the point A. Line DA intersects circle w_1 at the point S. Line O_1O_2 intersects line SC at the point F. E is an intersection point of circle w_1 and circumcircle w_3 of $\triangle ADF$. Prove that line O_1E tangents circle w_3

– Natural number $m \geq 2$ is given. There are n, $n \geq 3$ non-collinear points, pairwise connected by segments. Every segment is coloured one of the m given colours (there are segments of each colour) so that the following condition is true : if in a triangle two sides coloured the same colour, then the third side is also coloured this colour. What is the maximum value of n with the given m?

– Function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following equation for all real x:

$$f(f(x)) = x^2 f(x) - x + 1$$

. Find $f(1)$
