Belarusian National Olympiad 2009

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by proximo

– Day 1

– Let AB be a chord on parabola $y = x^2$ and $AB||Ox$. For each point C on parabola different from A and B we are taking point C_1 lying on the circumcircle of $\triangle ABC$ such that $CC_1||Oy$. Find a locus of points C_1.

– In the trapezoid $ABCD$, ($BC||AD$) $\angle BCD = 72^\circ$, $AD = BD = CD$. Let point K be a point on BD such that $AK = AD$. M is a midpoint of CD. N is an intersection point of AM and BD. Prove that $BK = ND$.

– Find all pairs (m, n) of natural numbers such that

$$m! + n! = m^n$$

– Two players are playing the game. On each turn player writes a pair of non-negative integer numbers (a, b), satisfying the condition that for each written earlier pair (c, d) $a < c$ or $b < d$. The play who write the pair $(0, 0)$ looses. Who of the players wins if he is playing correctly.

– Day 2

– In acute triangle $\triangle ABC$ $\angle C = 60^\circ$. Let B_1 and A_1 be the points on sides AC and BC respectively. Circumcircles of $\triangle BCB_1$ and $\triangle ACA_1$ intersect at the points C and D. Prove that D is a point on side AB if and only if $\frac{CB_1}{CB} + \frac{CA_1}{CA} = 1$

– Let $P(x), Q(x)$ be non-constant polynomials with integer coefficients. It is known that polynomial $P(x)Q(x) - 2009$ has at least 25 distinct integer roots. Prove that the degree of each polynomial $P(x)$ and $Q(x)$ is qreater than 2.

– The sum of the twenty distinct integer numbers equals 210.
a) Show that the sum of their squares is not less than 2870
b)Find these numbers if the sum of their squares equals 2870

– Let P be a non-empty set of natural numbers, which is with any pair of elements contains also their sum. The amount of elements of P, that cannot be represented as $n + x$, where $x \in P$ is called the *index* of element n (if this amount is a finite number, otherwise the index is counted as infinite).

a) Prove that the index of the sum of two elements equals the sum of the indices of these elements
b)Prove that the index of each element from the set P is finite.
c) Prove that the index of element $n \in P$ is not greater than n

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