China Team Selection Test 2016

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TST 1

Day 1

1 $ABCDEF$ is a cyclic hexagon with $AB = BC = CD = DE$. K is a point on segment AE satisfying $\angle BKC = \angle KFE, \angle CKD = \angle KFA$. Prove that $KC = KF$.

2 Find the smallest positive number λ , such that for any complex numbers $z_1, z_2, z_3 \in \{z \in C \big| |z| < 1\}$,if $z_1 + z_2 + z_3 = 0$, then

$$\left|z_1z_2 + z_2z_3 + z_3z_1\right|^2 + \left|z_1z_2z_3\right|^2 < \lambda.$$

3 Let $n \geq 2$ be a natural. Define

$$X = \{(a_1, a_2, \cdots, a_n) | a_k \in \{0, 1, 2, \cdots, k\}, k = 1, 2, \cdots, n\}$$

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For any two elements $s = (s_1, s_2, \cdots, s_n) \in X, t = (t_1, t_2, \cdots, t_n) \in X$, define

$$s \vee t = (\max\{s_1, t_1\}, \max\{s_2, t_2\}, \cdots, \max\{s_n, t_n\})$$

$$s \wedge t = (\min\{s_1, t_1\}, \min\{s_2, t_2, \}, \cdots, \min\{s_n, t_n\})$$

Find the largest possible size of a proper subset A of X such that for any $s, t \in A$, one has $s \vee t \in A, s \wedge t \in A$.

Day 2

4 Let $c, d \geq 2$ be naturals. Let $\{a_n\}$ be the sequence satisfying $a_1 = c, a_{n+1} = a_n^d + c$ for $n = 1, 2, \cdots$.
Prove that for any $n \geq 2$, there exists a prime number p such that $p | a_n$ and $p \not| a_i$ for $i = 1, 2, \cdots n - 1$.

5 Refer to the diagram below. Let $ABCD$ be a cyclic quadrilateral with center O. Let the internal angle bisectors of $\angle A$ and $\angle C$ intersect at I and let those of $\angle B$ and $\angle D$ intersect at J. Now extend AB and CD to intersect IJ and P and R respectively and let IJ intersect BC and DA at Q and S respectively. Let the midpoints of PR and QS be M and N respectively. Given that O does not lie on the line IJ, show that OM and ON are perpendicular.

6 Let m,n be naturals satisfying $n \ge m \ge 2$ and let S be a set consisting of n naturals. Prove that S has at least 2^{n-m+1} distinct subsets, each whose sum is divisible by m. (The zero set counts as a subset).

TST 2

Day 1

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1 P is a point in the interior of acute triangle ABC. D,E,F are the reflections of P across BC,CA,AB respectively. Rays AP,BP,CP meet the circumcircle of $\triangle ABC$ at L,M,N respectively. Prove that the circumcircles of $\triangle PDL, \triangle PEM, \triangle PFN$ meet at a point T different from P.

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2 Find the smallest positive number λ, such that for any 12 points on the plane $P_1,P_2,\ldots,P_{12}$(can overlap), if the distance between any two of them does not exceed 1, then $\sum_{1\le i<j\le 12} |P_iP_j|^2 \le \lambda$.

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3 Let P be a finite set of primes, A an infinite set of positive integers, where every element of A has a prime factor not in P. Prove that there exist an infinite subset B of A, such that the sum of elements in any finite subset of B has a prime factor not in P.

Day 2

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4 Set positive integer $m=2^k\cdot t$, where k is a non-negative integer, t is an odd number, and let $f(m)=t^{1-k}$. Prove that for any positive integer n and for any positive odd number $a\le n$, $\prod_{m=1}^n f(m)$ is a multiple of a.

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5 Does there exist two infinite positive integer sets S,T, such that any positive integer n can be uniquely expressed in the form

$$n = s_1t_1+s_2t_2+\ldots+s_kt_k$$

,where k is a positive integer dependent on n, $s_1<\ldots<s_k$ are elements of S, $t_1,\ldots,t_k$ are elements of T?

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6 The diagonals of a cyclic quadrilateral $ABCD$ intersect at P, and there exist a circle Γ tangent to the extensions of AB,BC,AD,DC at X,Y,Z,T respectively. Circle Ω passes through points A,B, and is externally tangent to circle Γ at S. Prove that $SP\perp ST$.

TST 3

Day 1

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1 Let n be an integer greater than 1, α is a real, $0 < \alpha < 2$, $a_1, \ldots, a_n, c_1, \ldots, c_n$ are all positive numbers. For $y > 0$, let

$$f(y) = \left(\sum_{a_i \le y} c_i a_i^2\right)^{\frac{1}{2}} + \left(\sum_{a_i > y} c_i a_i^{\alpha}\right)^{\frac{1}{\alpha}}.$$

If positive number x satisfies $x \ge f(y)$ (for some y), prove that $f(x) \le 8^{\frac{1}{\alpha}} \cdot x$.

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2 In the coordinate plane the points with both coordinates being rational numbers are called rational points. For any positive integer n, is there a way to use n colours to colour all rational points, every point is coloured one colour, such that any line segment with both endpoints being rational points contains the rational points of every colour?

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3 In cyclic quadrilateral $ABCD$, $AB > BC$, $AD > DC$, I, J are the incenters of $\triangle ABC$,$\triangle ADC$ respectively. The circle with diameter AC meets segment IB at X, and the extension of JD at Y. Prove that if the four points B, I, J, D are concyclic, then X, Y are the reflections of each other across AC.

Day 2

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4 Let a, b, b', c, m, q be positive integers, where $m > 1, q > 1, |b - b'| \ge a$. It is given that there exist a positive integer M such that

$$S_q(an + b) \equiv S_q(an + b') + c \pmod{m}$$

holds for all integers $n \ge M$. Prove that the above equation is true for all positive integers n. (Here $S_q(x)$ is the sum of digits of x taken in base q).

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5 Let S be a finite set of points on a plane, where no three points are collinear, and the convex hull of S, Ω, is a $2016-$gon $A_1A_2\ldots A_{2016}$. Every point on S is labelled one of the four numbers $\pm 1, \pm 2$, such that for $i = 1, 2, \ldots, 1008,$ the numbers labelled on points A_i and A_{i+1008} are the negative of each other.
Draw triangles whose vertices are in S, such that any two triangles do not have any common interior points, and the union of these triangles is Ω. Prove that there must exist a triangle, where the numbers labelled on some two of its vertices are the negative of each other.

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6 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition: for any three distinct real numbers a, b, c, a triangle can be formed with side lengths a, b, c, if and only if a triangle can be formed with side lengths $f(a), f(b), f(c)$.
