**Problems from the CMIMC 2016**

www.artofproblemsolving.com/community/c253587

by djmathman

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|---|---|
| – | Algebra |
| 1 | In a race, people rode either bicycles with blue wheels or tricycles with tan wheels. Given that 15 more people rode bicycles than tricycles and there were 15 more tan wheels than blue wheels, what is the total number of people who rode in the race? |
| 2 | Suppose that some real number $x$ satisfies $$\log_2 x + \log_8 x + \log_{64} x = \log_x 2 + \log_x 16 + \log_x 128.$$ Given that the value of $\log_2 x + \log_x 2$ can be expressed as $\frac{a\sqrt{b}}{c}$, where $a$ and $c$ are coprime positive integers and $b$ is squarefree, compute $abc$. |
| 3 | Let $\ell$ be a real number satisfying the equation $\frac{(1+\ell)^2}{1+\ell^2} = \frac{13}{37}$. Then $$\frac{(1+\ell)^3}{1+\ell^3} = \frac{m}{n},$$ where $m$ and $n$ are positive coprime integers. Find $m+n$. |
| 4 | A line with negative slope passing through the point $(18,8)$ intersects the $x$ and $y$ axes at $(a,0)$ and $(0,b)$, respectively. What is the smallest possible value of $a+b$? |
| 5 | The parabolas $y = x^2 + 15x + 32$ and $x = y^2 + 49y + 593$ meet at one point $(x_0, y_0)$. Find $x_0 + y_0$. |
| 6 | For some complex number $\omega$ with $\lvert\omega\rvert = 2016$, there is some real $\lambda > 1$ such that $\omega, \omega^2$, and $\lambda\omega$ form an equilateral triangle in the complex plane. Then, $\lambda$ can be written in the form $\frac{a+\sqrt{b}}{c}$, where $a, b,$ and $c$ are positive integers and $b$ is squarefree. Compute $\sqrt{a+b+c}$. |
| 7 | Suppose $a$, $b$, $c$, and $d$ are positive real numbers that satisfy the system of equations $$(a+b)(c+d) = 143,$$ $$(a+c)(b+d) = 150,$$ $$(a+d)(b+c) = 169.$$ Compute the smallest possible value of $a^2+b^2+c^2+d^2$. |

**8** Let $r_1, r_2, \ldots, r_{20}$ be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$

can be written in the form $\frac{m}{n}$ where $m$ and $n$ are positive coprime integers, find $m+n$.

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**9** Let $\lfloor x \rfloor$ denote the greatest integer function and $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of $x$. Let $1 \leq x_1 < \ldots < x_{100}$ be the $100$ smallest values of $x \geq 1$ such that $\sqrt{\lfloor x \rfloor \lfloor x^3 \rfloor} + \sqrt{\{x\}\{x^3\}} = x^2$. Compute

$$\sum_{k=1}^{50} \frac{1}{x_{2k}^2 - x_{2k-1}^2}.$$

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**10** Denote by $F_0(x), F_1(x), \ldots$ the sequence of Fibonacci polynomials, which satisfy the recurrence $F_0(x) = 1$, $F_1(x) = x$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for all $n \geq 2$. It is given that there exist unique integers $\lambda_0, \lambda_1, \ldots, \lambda_{1000}$ such that

$$x^{1000} = \sum_{i=0}^{1000} \lambda_i F_i(x)$$

for all real $x$. For which integer $k$ is $|\lambda_k|$ maximized?

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**–** Combinatorics

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**1** The phrase "COLORFUL TARTAN" is spelled out with wooden blocks, where blocks of the same letter are indistinguishable. How many ways are there to distribute the blocks among two bags of different color such that neither bag contains more than one of the same letter?

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**2** Six people each flip a fair coin. Everyone who flipped tails then flips their coin again. Given that the probability that all the coins are now heads can be expressed as simplified fraction $\frac{m}{n}$, compute $m+n$.

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**3** At CMU, markers come in two colors: blue and orange. Zachary fills a hat randomly with three markers such that each color is chosen with equal probability, then Chase shuffles an additional orange marker into the hat. If Zachary chooses one of the markers in the hat at random and it turns out to be orange, the probability that there is a second orange marker in the hat can be expressed as simplified fraction $\frac{m}{n}$. Find $m+n$.

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**4** Kevin colors three distinct squares in a $3 \times 3$ grid red. Given that there exist two uncolored squares such that coloring one of them would create a horizontal or vertical red line, find the number of ways he could have colored the original three squares.

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**5** Let $\mathcal{S}$ be a regular 18-gon, and for two vertices in $\mathcal{S}$ define the *distance* between them to be the length of the shortest path along the edges of $\mathcal{S}$ between them (e.g. adjacent vertices have distance 1). Find the number of ways to choose three distinct vertices from $\mathcal{S}$ such that no two of them have distance 1, 8, or 9.

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**6** Shen, Ling, and Ru each place four slips of paper with their name on it into a bucket. They then play the following game: slips are removed one at a time, and whoever has all of their slips removed first wins. Shen cheats, however, and adds an extra slip of paper into the bucket, and will win when four of his are drawn. Given that the probability that Shen wins can be expressed as simplified fraction $\tfrac{m}{n}$, compute $m+n$.

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**7** There are eight people, each with their own horse. The horses are arbitrarily arranged in a line from left to right, while the people are lined up in random order to the left of all the horses. One at a time, each person moves rightwards in an attempt to reach their horse. If they encounter a mounted horse on their way to their horse, the mounted horse shouts angrily at the person, who then scurries home immediately. Otherwise, they get to their horse safely and mount it. The expected number of people who have scurried home after all eight people have attempted to reach their horse can be expressed as simplified fraction $\tfrac{m}{n}$. Find $m+n$.

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**8** Brice is eating bowls of rice. He takes a random amount of time $t_1 \in (0,1)$ minutes to consume his first bowl, and every bowl thereafter takes $t_n = t_{n-1} + r_n$ minutes, where $t_{n-1}$ is the time it took him to eat his previous bowl and $r_n \in (0,1)$ is chosen uniformly and randomly. The probability that it takes Brice at least 12 minutes to eat 5 bowls of rice can be expressed as simplified fraction $\tfrac{m}{n}$. Compute $m+n$.

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**9** 1007 distinct potatoes are chosen independently and randomly from a box of 2016 potatoes numbered $1, 2, \ldots, 2016$, with $p$ being the smallest chosen potato. Then, potatoes are drawn one at a time from the remaining 1009 until the first one with value $q < p$ is drawn. If no such $q$ exists, let $S = 1$. Otherwise, let $S = pq$. Then given that the expected value of $S$ can be expressed as simplified fraction $\tfrac{m}{n}$, find $m+n$.

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**10** For all positive integers $m \geq 1$, denote by $\mathcal{G}_m$ the set of simple graphs with exactly $m$ edges. Find the number of pairs of integers $(m,n)$ with $1 < 2n \leq m \leq 100$ such that there exists a simple graph $G \in \mathcal{G}_m$ satisfying the following property: it is possible to label the edges of $G$ with labels $E_1$, $E_2$, $\ldots$, $E_m$ such that for all $i \neq j$, edges $E_i$ and $E_j$ are incident if and only if either $|i-j| \leq n$ or $|i-j| \geq m-n$.

*Note:* A graph is said to be *simple* if it has no self-loops or multiple edges. In other words, no edge connects a vertex to itself, and the number of edges connecting two distinct vertices is either $0$ or $1$.

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**–** Computer Science

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**1** For how many distinct ordered triples $(a, b, c)$ of boolean variables does the expression $a \vee (b \wedge c)$ evaluate to true?

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**2** In concurrent computing, two processes may have their steps interwoven in an unknown order, as long as the steps of each process occur in order. Consider the following two processes:

| Process | $A$ | $B$ |
|---|---|---|
| Step 1 | $x \leftarrow x - 4$ | $x \leftarrow x - 5$ |
| Step 2 | $x \leftarrow x \cdot 3$ | $x \leftarrow x \cdot 4$ |
| Step 3 | $x \leftarrow x - 4$ | $x \leftarrow x - 5$ |
| Step 4 | $x \leftarrow x \cdot 3$ | $x \leftarrow x \cdot 4$ |

One such interweaving is $A1$, $B1$, $A2$, $B2$, $A3$, $B3$, $B4$, $A4$, but $A1$, $A3$, $A2$, $A4$, $B1$, $B2$, $B3$, $B4$ is not since the steps of $A$ do not occur in order. We run $A$ and $B$ concurrently with $x$ initially valued at $6$. Find the minimal possible value of $x$ among all interweavings.

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**3** Sophia writes an algorithm to solve the graph isomorphism problem. Given a graph $G = (V, E)$, her algorithm iterates through all permutations of the set $\{v_1, \dots, v_{|V|}\}$, each time examining all ordered pairs $(v_i, v_j) \in V \times V$ to see if an edge exists. When $|V| = 8$, her algorithm makes $N$ such examinations. What is the largest power of two that divides $N$?

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**4** Given a list $A$, let $f(A) = [A[0] + A[1], A[0] - A[1]]$. Alef makes two programs to compute $f(f(...(f(A))))$, where the function is composed $n$ times:

```
FUNCTION T1(A, n)
    IF n = 0
        RETURN A
    ELSE
        RETURN [T1(A, n-1)[0] + T1(A, n-1)[1],
         T1(A, n-1)[0] - T1(A, n-1)[1]]
```

```
FUNCTION T2(A, n)
    IF n = 0
        RETURN A
    ELSE
        B <- T2(A, n-1)
        RETURN [B[0] + B[1], B[0] - B[1]]
```

Each time $T_1$ or $T_2$ is called, Alef has to pay one dollar. How much money does he save by calling $T_2([13, 37], 4)$ instead of $T_1([13, 37], 4)$?

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**5** We define the *weight* of a path to be the sum of the numbers written on each edge of the path. Find the minimum weight among all paths in the graph below that visit each vertex precisely once:

`http://i.imgur.com/V99Eg9j.png`

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**6** Aaron is trying to write a program to compute the terms of the sequence defined recursively by $a_0 = 0$, $a_1 = 1$, and

$$a_n = \begin{cases} a_{n-1} - a_{n-2} & n \equiv 0 \pmod 2 \\ 2a_{n-1} - a_{n-2} & \text{else} \end{cases}$$

However, Aaron makes a typo, accidentally computing the recurrence by

$$a_n = \begin{cases} a_{n-1} - a_{n-2} & n \equiv 0 \pmod 3 \\ 2a_{n-1} - a_{n-2} & \text{else} \end{cases}$$

For how many $0 \le k \le 2016$ did Aaron coincidentally compute the correct value of $a_k$?

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**7** Given the list

$$A = [9, 12, 1, 20, 17, 4, 10, 7, 15, 8, 13, 14],$$

we would like to sort it in increasing order. To accomplish this, we will perform the following operation repeatedly: remove an element, then insert it at any position in the list, shifting elements if necessary. What is the minimum number of applications of this operation necessary to sort $A$?

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**8** Consider the sequence of sets defined by $S_0 = \{0, 1\}, S_1 = \{0, 1, 2\}$, and for $n \ge 2$,

$$S_n = S_{n-1} \cup \{2^n + x \mid x \in S_{n-2}\}.$$

For example, $S_2 = \{0, 1, 2\} \cup \{2^2 + 0, 2^2 + 1\} = \{0, 1, 2, 4, 5\}$. Find the $200$th smallest element of $S_{2016}$.

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**9** Ryan has three distinct eggs, one of which is made of rubber and thus cannot break; unfortunately, he doesn't know which egg is the rubber one. Further, in some 100-story building there exists a floor such that all normal eggs dropped from below that floor will not break, while those dropped from at or above that floor will break and cannot be dropped again. What is the minimum number of times Ryan must drop an egg to determine the floor satisfying this property?

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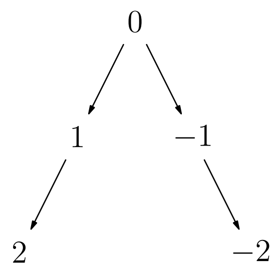
**10** Given $x_0 \in \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$, we define the *non-redundant binary tree* $T(x_0, f, g)$ in the following way:

-The tree $T$ initially consists of just $x_0$ at height $0$.

-Let $v_0, \dots, v_k$ be the vertices at height $h$. Then the vertices of height $h+1$ are added to $T$ by: for $i = 0, 1, \dots, k$, $f(v_i)$ is added as a child of $v_i$ if $f(v_i) \notin T$, and $g(v_i)$ is added as a child of $v_i$ if $g(v_i) \notin T$.

For example, if $f(x) = x + 1$ and $g(x) = x - 1$, then the first three layers of $T(0, f, g)$ look like:




If $f(x) = 1024x - 2047\lfloor x/2 \rfloor$ and $g(x) = 2x - 3\lfloor x/2 \rfloor + 2\lfloor x/4 \rfloor$, then how many vertices are in $T(2016, f, g)$?

**–** Geometry

**1** Let $\triangle ABC$ be an equilateral triangle and $P$ a point on $\overline{BC}$. If $PB = 50$ and $PC = 30$, compute $PA$.

**2** Let $ABCD$ be an isosceles trapezoid with $AD = BC = 15$ such that the distance between its bases $AB$ and $CD$ is $7$. Suppose further that the circles with diameters $\overline{AD}$ and $\overline{BC}$ are tangent to each other. What is the area of the trapezoid?

**3** Let $ABC$ be a triangle. The angle bisector of $\angle B$ intersects $AC$ at point $P$, while the angle bisector of $\angle C$ intersects $AB$ at a point $Q$. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is $72$. The length of $\overline{BC}$ can be written in the form $m\sqrt{n}$ where $m$ and $n$ are positive integers with $n$ as small as possible. What is $m + n$?

**4** Andrew the Antelope canters along the surface of a regular icosahedron, which has twenty equilateral triangle faces and edge length 4. If he wants to move from one vertex to the opposite vertex, the minimum distance he must travel can be expressed as $\sqrt{n}$ for some integer $n$. Compute $n$.

**5** Let $\mathcal{P}$ be a parallelepiped with side lengths $x$, $y$, and $z$. Suppose that the four space diagonals of $\mathcal{P}$ have lengths $15$, $17$, $21$, and $23$. Compute $x^2 + y^2 + z^2$.

**6** In parallelogram $ABCD$, angles $B$ and $D$ are acute while angles $A$ and $C$ are obtuse. The perpendicular from $C$ to $AB$ and the perpendicular from $A$ to $BC$ intersect at a point $P$ inside the parallelogram. If $PB = 700$ and $PD = 821$, what is $AC$?

**7** Let $ABC$ be a triangle with incenter $I$ and incircle $\omega$. It is given that there exist points $X$ and $Y$ on the circumference of $\omega$ such that $\angle BXC = \angle BYC = 90^\circ$. Suppose further that $X$, $I$, and $Y$ are collinear. If $AB = 80$ and $AC = 97$, compute the length of $BC$.

**8** Suppose $ABCD$ is a convex quadrilateral satisfying $AB = BC$, $AC = BD$, $\angle ABD = 80^\circ$, and $\angle CBD = 20^\circ$. What is $\angle BCD$ in degrees?

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**9** Let $\triangle ABC$ be a triangle with $AB = 65$, $BC = 70$, and $CA = 75$. A semicircle $\Gamma$ with diameter $\overline{BC}$ is constructed outside the triangle. Suppose there exists a circle $\omega$ tangent to $AB$ and $AC$ and furthermore internally tangent to $\Gamma$ at a point $X$. The length $AX$ can be written in the form $m\sqrt{n}$ where $m$ and $n$ are positive integers with $n$ not divisible by the square of any prime. Find $m + n$.

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**10** Let $\triangle ABC$ be a triangle with circumcircle $\Omega$ and let $N$ be the midpoint of the major arc $\widehat{BC}$. The incircle $\omega$ of $\triangle ABC$ is tangent to $AC$ and $AB$ at points $E$ and $F$ respectively. Suppose point $X$ is placed on the same side of $EF$ as $A$ such that $\triangle XEF \sim \triangle ABC$. Let $NX$ intersect $BC$ at a point $P$. Given that $AB = 15$, $BC = 16$, and $CA = 17$, compute $\frac{PX}{XN}$.

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**–** Number Theory

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**1** David, when submitting a problem for CMIMC, wrote his answer as $100\frac{x}{y}$, where $x$ and $y$ are two positive integers with $x < y$. Andrew interpreted the expression as a product of two rational numbers, while Patrick interpreted the answer as a mixed fraction. In this case, Patrick's number was exactly double Andrew's! What is the smallest possible value of $x + y$?

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**2** Let $a_1, a_2, \ldots$ be an infinite sequence of (positive) integers such that $k$ divides $\gcd(a_{k-1}, a_k)$ for all $k \geq 2$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_{10}$.

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**3** How many pairs of integers $(a, b)$ are there such that $0 \leq a < b \leq 100$ and such that $\frac{2^b - 2^a}{2016}$ is an integer?

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**4** For some positive integer $n$, consider the usual prime factorization

$$n = \prod_{i=1}^{k} p_i^{e_i} = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k},$$

where $k$ is the number of primes factors of $n$ and $p_i$ are the prime factors of $n$. Define $Q(n), R(n)$ by

$$Q(n) = \prod_{i=1}^{k} p_i^{p_i} \text{ and } R(n) = \prod_{i=1}^{k} e_i^{e_i}.$$

For how many $1 \leq n \leq 70$ does $R(n)$ divide $Q(n)$?

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**5** Determine the sum of the positive integers $n$ such that there exist primes $p, q, r$ satisfying $p^n + q^2 = r^2$.

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**6** Define a *tasty residue* of $n$ to be an integer $1 < a < n$ such that there exists an integer $m > 1$ satisfying

$$a^m \equiv a \pmod{n}.$$

Find the number of tasty residues of $2016$.

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**7** Determine the smallest positive prime $p$ which satisfies the congruence

$$p + p^{-1} \equiv 25 \pmod{143}.$$

Here, $p^{-1}$ as usual denotes multiplicative inverse.

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**8** Given that

$$\sum_{x=1}^{70}\sum_{y=1}^{70}\frac{x^y}{y} = \frac{m}{67!}$$

for some positive integer $m$, find $m \pmod{71}$.

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**9** Compute the number of positive integers $n \le 50$ such that there exist distinct positive integers $a, b$ satisfying

$$\frac{a}{b} + \frac{b}{a} = n\left(\frac{1}{a} + \frac{1}{b}\right).$$

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**10** Let $f : \mathbb{N} \mapsto \mathbb{R}$ be the function

$$f(n) = \sum_{k=1}^{\infty}\frac{1}{\operatorname{lcm}(k, n)^2}.$$

It is well-known that $f(1) = \frac{\pi^2}{6}$. What is the smallest positive integer $m$ such that $m \cdot f(10)$ is the square of a rational multiple of $\pi$?

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**–** Team

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**1** Construction Mayhem University has been on a mission to expand and improve its campus! The university has recently adopted a new construction schedule where a new project begins every two days. Each project will take exactly one more day than the previous one to complete (so the first project takes 3, the second takes 4, and so on.)

Suppose the new schedule starts on Day 1. On which day will there first be at least $10$ projects in place at the same time?

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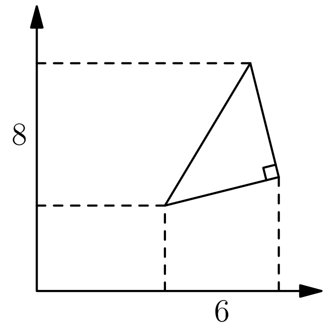
**2** Right isosceles triangle $T$ is placed in the first quadrant of the coordinate plane. Suppose that the projection of $T$ onto the $x$-axis has length $6$, while the projection of $T$ onto the $y$-axis has length $8$. What is the sum of all possible areas of the triangle $T$?

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**3** We have 7 buckets labelled 0-6. Initially bucket 0 is empty, while bucket $n$ (for each $1 \leq n \leq 6$) contains the list $[1, 2, \dots, n]$. Consider the following program: choose a subset $S$ of $[1, 2, \dots, 6]$ uniformly at random, and replace the contents of bucket $|S|$ with $S$. Let $\frac{p}{q}$ be the probability that bucket 5 still contains $[1, 2, \dots, 5]$ after two executions of this program, where $p, q$ are positive coprime integers. Find $p$.

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**4** For some integer $n > 0$, a square paper of side length $2^n$ is repeatedly folded in half, right-to-left then bottom-to-top, until a square of side length 1 is formed. A hole is then drilled into the square at a point $\frac{3}{16}$ from the top and left edges, and then the paper is completely unfolded. The holes in the unfolded paper form a rectangular array of unevenly spaced points; when connected with horizontal and vertical line segments, these points form a grid of squares and rectangles. Let $P$ be a point chosen randomly from *inside* this grid. Suppose the largest $L$ such that, for all $n$, the probability that the four segments $P$ is bounded by form a square is at least $L$ can be written in the form $\frac{m}{n}$ where $m$ and $n$ are positive relatively prime integers. Find $m+n$.

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**5** Recall that in any row of Pascal's Triangle, the first and last elements of the row are $1$ and each other element in the row is the sum of the two elements above it from the previous row. With this in mind, define the *Pascal Squared Triangle* as follows:

- In the $n^{\text{th}}$ row, where $n \geq 1$, the first and last elements of the row equal $n^2$;
- Each other element is the sum of the two elements directly above it.

The first few rows of the Pascal Squared Triangle are shown below.

| | |
|---|---|
| Row 1: | $1$ |
| Row 2: | $4 \quad 4$ |
| Row 3: | $9 \quad 8 \quad 9$ |
| Row 4: | $16 \quad 17 \quad 17 \quad 16$ |
| Row 5: | $25 \quad 33 \quad 34 \quad 33 \quad 25$ |

Let $S_n$ denote the sum of the entries in the $n^{\text{th}}$ row. For how many integers $1 \leq n \leq 10^6$ is $S_n$ divisible by $13$?

**6** Suppose integers $a < b < c$ satisfy

$$a + b + c = 95 \qquad \text{and} \qquad a^2 + b^2 + c^2 = 3083.$$

Find $c$.

**7** In $\triangle ABC$, $AB = 17$, $AC = 25$, and $BC = 28$. Points $M$ and $N$ are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively, and $P$ is a point on $\overline{BC}$. Let $Q$ be the second intersection point of the circumcircles of $\triangle BMP$ and $\triangle CNP$. It is known that as $P$ moves along $\overline{BC}$, line $PQ$ passes through some fixed point $X$. Compute the sum of the squares of the distances from $X$ to each of $A$, $B$, and $C$.

**8** Let $N$ be the number of triples of positive integers $(a, b, c)$ with $a \leq b \leq c \leq 100$ such that the polynomial

$$P(x) = x^2 + (a^2 + 4b^2 + c^2 + 1)x + (4ab + 4bc - 2ca)$$

has integer roots in $x$. Find the last three digits of $N$.

**9** For how many permutations $\pi$ of $\{1, 2, \ldots, 9\}$ does there exist an integer $N$ such that

$$N \equiv \pi(i) \pmod{i} \text{ for all integers } 1 \leq i \leq 9?$$

**10** Let $\mathcal{P}$ be the unique parabola in the $xy$-plane which is tangent to the $x$-axis at $(5,0)$ and to the $y$-axis at $(0,12)$. We say a line $\ell$ is $\mathcal{P}$-friendly if the $x$-axis, $y$-axis, and $\mathcal{P}$ divide $\ell$ into three segments, each of which has equal length. If the sum of the slopes of all $\mathcal{P}$-friendly lines can be written in the form $-\tfrac{m}{n}$ for $m$ and $n$ positive relatively prime integers, find $m + n$.

**–** Algebra Tiebreaker

**1** Let

$$f(x) = \frac{1}{1 - \frac{1}{1 - x}}.$$

Compute $f^{2016}(2016)$, where $f$ is composed upon itself $2016$ times.

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**2** Determine the value of the sum

$$\left|\sum_{1\le i<j\le 50} ij(-1)^{i+j}\right|.$$

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**3** Suppose $x$ and $y$ are real numbers which satisfy the system of equations

$$x^2-3y^2=\frac{17}{x} \qquad \text{and} \qquad 3x^2-y^2=\frac{23}{y}.$$

Then $x^2+y^2$ can be written in the form $\sqrt[m]{n}$, where $m$ and $n$ are positive integers and $m$ is as small as possible. Find $m+n$.

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**–** Combinatorics Tiebreaker

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**1** For a set $S\subseteq\mathbb{N}$, define $f(S)=\{\lceil\sqrt{s}\rceil \mid s\in S\}$. Find the number of sets $T$ such that $|f(T)|=2$ and $f(f(T))=\{2\}$.

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**2** Let $S=\{1,2,3,4,5,6,7\}$. Compute the number of sets of subsets $T=\{A,B,C\}$ with $A,B,C\in S$ such that $A\cup B\cup C=S$, $(A\cap C)\cup(B\cap C)=\emptyset$, and no subset contains two consecutive integers.

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**3** Let $S$ be the set containing all positive integers whose decimal representations contain only 3s and 7s, have at most 1998 digits, and have at least one digit appear exactly 999 times. If $N$ denotes the number of elements in $S$, find the remainder when $N$ is divided by 1000.

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**–** Computer Science Tiebreaker

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**1** A *planar* graph is a connected graph that can be drawn on a sphere without edge crossings. Such a drawing will divide the sphere into a number of faces. Let $G$ be a planar graph with $11$ vertices of degree $2$, $5$ vertices of degree $3$, and $1$ vertex of degree $7$. Find the number of faces into which $G$ divides the sphere.

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**2** The *Stooge sort* is a particularly inefficient recursive sorting algorithm defined as follows: given an array $A$ of size $n$, we swap the first and last elements if they are out of order; we then (if $n\ge 3$) Stooge sort the first $\lceil\frac{2n}{3}\rceil$ elements, then the last $\lceil\frac{2n}{3}\rceil$, then the first $\lceil\frac{2n}{3}\rceil$ elements again. Given that this runs in $O(n^\alpha)$, where $\alpha$ is minimal, find the value of $(243/32)^\alpha$.

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**3** Let $\varepsilon$ denote the empty string. Given a pair of strings $(A,B)\in\{0,1,2\}^*\times\{0,1\}^*$, we are allowed the following operations:

$$\begin{cases} (A,1) \to (A0,\varepsilon) \\ (A,10) \to (A00,\varepsilon) \\ (A,0B) \to (A0,B) \\ (A,11B) \to (A01,B) \\ (A,100B) \to (A0012,1B) \\ (A,101B) \to (A00122,10B) \end{cases}$$

We perform these operations on $(A,B)$ until we can no longer perform any of them. We then iteratively delete any instance of $20$ in $A$ and replace any instance of $21$ with $1$ until there are no such substrings remaining. Among all binary strings $X$ of size $9$, how many different possible outcomes are there for this process performed on $(\varepsilon,X)$?

– Geometry Tiebreaker

1 Point $A$ lies on the circumference of a circle $\Omega$ with radius $78$. Point $B$ is placed such that $AB$ is tangent to the circle and $AB=65$, while point $C$ is located on $\Omega$ such that $BC=25$. Compute the length of $\overline{AC}$.

2 Identical spherical tennis balls of radius 1 are placed inside a cylindrical container of radius 2 and height 19. Compute the maximum number of tennis balls that can fit entirely inside this container.

3 Triangle $ABC$ satisfies $AB=28$, $BC=32$, and $CA=36$, and $M$ and $N$ are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Let point $P$ be the unique point in the plane $ABC$ such that $\triangle PBM \sim \triangle PNC$. What is $AP$?

– Number Theory Tiebreaker

1 For all integers $n\geq 2$, let $f(n)$ denote the largest positive integer $m$ such that $\sqrt[m]{n}$ is an integer. Evaluate

$$f(2)+f(3)+\cdots+f(100).$$

2 For each integer $n\geq 1$, let $S_n$ be the set of integers $k>n$ such that $k$ divides $30n-1$. How many elements of the set

$$\mathcal{S}=\bigcup_{i\geq 1} S_i = S_1\cup S_2\cup S_3\cup\ldots$$

are less than $2016$?

**3** Let $\{x\}$ denote the fractional part of $x$. For example, $\{5.5\} = 0.5$. Find the smallest prime $p$ such that the inequality

$$\sum_{n=1}^{p^2} \left\{ \frac{n^p}{p^2} \right\} > 2016$$

holds.