IMC 2015

www.artofproblemsolving.com/community/c254975

by randomusername, Gza Ks

– Day 1

---

1 For any integer $n \geq 2$ and two $n \times n$ matrices with real entries $A,\ B$ that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(Proposed by Zbigniew Skoczylas, Wroclaw University of Technology)

---

2 For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

(Proposed by Stephan Wagner, Stellenbosch University)

---

3 Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$.

Determine whether or not $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$ is a rational number.

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

---

4 Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16). \qquad (1)$$

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

---

5 Let $n \geq 2$, let $A_1, A_2, \ldots, A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of $A_1, A_2, \ldots, A_{n+1}$. Prove that $\angle A_i B A_j > 90^\circ$ holds for at least n pairs (i, j) with $1 \leq i < j \leq n+1$.

Proposed by Gza Ks, Etvs University, Budapest

– Day 2

6 Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < 2.$$

Proposed by Ivan Krijan, University of Zagreb

7 Compute

$$\lim_{A \to +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}}\, dx.$$

Proposed by Jan ustek, University of Ostrava

8 Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

Proposed by Fedor Petrov, St. Petersburg State University

9 An $n \times n$ complex matrix A is called *t-normal* if $AA^t = A^t A$ where A^t is the transpose of A. For each n,
determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

Proposed by Shachar Carmeli, Weizmann Institute of Science

10 Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0 \leq x \leq 1} |p(x)| > \frac{1}{e^n}.$$

Proposed by Gza Ks, Etvs University, Budapest