8th RMM 2016

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Day 1 February 26, 2016

1 Let ABC be a triangle and let D be a point on the segment $BC, D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

2 Given positive integers m and $n \geq m$, determine the largest number of dominoes (1×2 or 2×1 rectangles) that can be placed on a rectangular board with m rows and $2n$ columns consisting of cells (1×1
squares) so that:
(i) each domino covers exactly two adjacent cells of the board;
(ii) no two dominoes overlap;
(iii) no two form a 2×2 square; and
(iv) the bottom row of the board is completely covered by n dominoes.

3 A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers. **(a)** Show that there exists a cubic sequence such that the only terms
of the sequence which are squares of integers are a_{2015} and a_{2016}. **(b)** Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence
satisfying the condition in part **(a)**.

Day 2 February 27, 2016

4 Let x and y be positive real numbers such that: $x + y^{2016} \geq 1$. Prove that $x^{2016} + y > 1 - \frac{1}{100}$

5 A convex hexagon $A_1B_1A_2B_2A_3B_3$ it is inscribed in a circumference Ω with radius R. The diagonals A_1B_2, A_2B_3, A_3B_1 are concurrent in X. For each $i = 1, 2, 3$ let ω_i tangent to the segments XA_i and XB_i and tangent to the arc A_iB_i of Ω that does not contain the other vertices of the hexagon; let r_i the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$ (b) If $R = r_1 + r_2 + r_3$, prove that the six points of tangency of the circumferences ω_i with the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic

6 A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-tree is a configuration of $n-1$ segments, each of which has an endpoint in $\mathcal{A}$ and an endpoint in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2, and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $|A_1B_1|+|A_2B_2|>|A_1B_2|+|A_2B_1|$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.