**2021 United States Ersatz Math Olympiad, 30-31 October 2021**

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**Day 1** October 30, 2021

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**1** Let $n$ be a fixed positive integer and consider an $n \times n$ grid of real numbers. Determine the greatest possible number of cells $c$ in the grid such that the entry in $c$ is both strictly greater than the average of $c$'s column and strictly less than the average of $c$'s row.

*Proposed by Holden Mui*

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**2** Find all integers $n \geq 1$ such that $2^n - 1$ has exactly $n$ positive integer divisors.

*Proposed by Ankan Bhattacharya*

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**3** Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let $O_1$ and $H_1$ denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let $O_2$ and $H_2$ denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines $O_1O_2$ and $H_1H_2$ are either parallel or coincide.

*Ankan Bhattacharya*

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**Day 2** October 31, 2021

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**4** Let $ABC$ be a triangle with circumcircle $\omega$, and let $X$ be the reflection of $A$ in $B$. Line $CX$ meets $\omega$ again at $D$. Lines $BD$ and $AC$ meet at $E$, and lines $AD$ and $BC$ meet at $F$. Let $M$ and $N$ denote the midpoints of $AB$ and $AC$.
Can line $EF$ share a point with the circumcircle of triangle $AMN$?

*Proposed by Sayandeep Shee*

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**5** Given a polynomial $p(x)$ with real coefficients, we denote by $S(p)$ the sum of the squares of its coefficients. For example $S(20x + 21) = 20^2 + 21^2 = 841.$
Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the indentity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2$$

*Proposed by Bhavya Tiwari*

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**6** A bagel is a loop of $2a + 2b + 4$ unit squares which can be obtained by cutting a concentric $a \times b$ hole out of an $(a + 2) \times (b + 2)$ rectangle, for some positive integers a and b. (The side of length a of the hole is parallel to the side of length $a + 2$ of the rectangle.)

Consider an infinite grid of unit square cells. For each even integer $n \geq 8$, a bakery of order $n$ is a finite set of cells $S$ such that, for every $n$-cell bagel $B$ in the grid, there exists a congruent copy of $B$ all of whose cells are in $S$. (The copy can be translated and rotated.) We denote by $f(n)$ the smallest possible number of cells in a bakery of order $n$.
Find a real number $\alpha$ such that, for all sufficiently large even integers $n \geq 8$, we have

$$\frac{1}{100} < \frac{f(n)}{n^\alpha} < 100$$

*Proposed by Nikolai Beluhov*