Math Majors of America Tournament for High Schools 2021

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by parmenides51, wu2481632

– Individual

1 Suppose that $20^{21} = 2^a5^b = 4^c5^d = 8^e5^f$ for positive integers $a, b, c, d, e,$ and f. Find $\frac{100bdf}{ace}$.

Proposed by Andrew Wu

2 Define the *digital reduction* of a two-digit positive integer $\underline{AB}$ to be the quantity $\underline{AB} - A - B$. Find the greatest common divisor of the digital reductions of all the two-digit positive integers. (For example, the digital reduction of 62 is $62 - 6 - 2 = 54$.)

Proposed by Andrew Wu

3 Let $ABCDEF$ be a regular hexagon with sidelength 6, and construct squares $ABGH$, $BCIJ$, $CDKL$, $DEMN$, $EFOP$, and $FAQR$ outside the hexagon. Find the perimeter of dodecagon $HGJILKNMPORQ$.

Proposed by Andrew Wu

4 Cat and Claire are having a conversation about Cat's favorite number. Cat says, "My favorite number is a two-digit positive prime integer whose first digit is less than its second, and when you reverse its digits, it's still a prime number!"

Claire asks, "If you picked a digit of your favorite number at random and revealed it to me without telling me which place it was in, is there any chance I'd know for certain what it is?"

Cat says, "Nope! However, if I now told you the units digit of my favorite number, you'd know which one it is!"

Claire says, "Now I know your favorite number!" What is Cat's favorite number?

Proposed by Andrew Wu

5 Suppose that $a_1 = 1$, and that for all $n \geq 2$, $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} + \ldots + (n-1)a_1$. Suppose furthermore that $b_n = a_1 + a_2 + \ldots + a_n$ for all n. If $b_1 + b_2 + b_3 + \ldots + b_{2021} = a_k$ for some k, find k.

Proposed by Andrew Wu

6 Ben flips a coin 10 times and then records the absolute difference between the total number of heads and tails he's flipped. He then flips the coin one more time and records the absolute difference between the total number of heads and tails he's flipped. If the probability that the second number he records is greater than the first can be expressed as $\frac{a}{b}$ for positive integers a, b with $\gcd(a, b) = 1$, then find $a + b$.

Proposed by Vismay Sharan

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7 Let $P_k(x) = (x-k)(x-(k+1))$. Kara picks four distinct polynomials from the set $\{P_1(x), P_2(x), P_3(x), \ldots, P_{12}(x)\}$ and discovers that when she computes the six sums of pairs of chosen polynomials, exactly two of the sums have two (not necessarily distinct) integer roots! How many possible combinations of four polynomials could Kara have picked?

Proposed by Andrew Wu

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8 Consider a hexagon with vertices labeled M, M, A, T, H, S in that order. Clayton starts at the M adjacent to M and A, and writes the letter down. Each second, Clayton moves to an adjacent vertex, each with probability $\frac{1}{2}$, and writes down the corresponding letter. Clayton stops moving when the string he's written down contains the letters M, A, T, and H in that order, not necessarily consecutively (for example, one valid string might be $MAMMSHTH$.) What is the expected length of the string Clayton wrote?

Proposed by Andrew Milas and Andrew Wu

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9 Suppose that $P(x)$ is a monic cubic polynomial with integer roots, and suppose that $\frac{P(a)}{a}$ is an integer for exactly 6 integer values of a. Suppose furthermore that exactly one of the distinct numbers $\frac{P(1)+P(-1)}{2}$ and $\frac{P(1)-P(-1)}{2}$ is a perfect square. Given that $P(0) > 0$, find the second-smallest possible value of $P(0)$.

Proposed by Andrew Wu

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10 Let ABC be a triangle with circumcenter O and incenter I, and suppose that OI meets AB and AC at P and Q, respectively. There exists a point R on arc $\widehat{BAC}$ such that the circumcircles of triangles PQR and ABC are tangent. Given that $AB = 14$, $BC = 20$, and $CA = 26$, find $\frac{RC}{RB}$.

Proposed by Andrew Wu

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11 If $\prod_{i=6}^{2021} (1 - \tan^2((2^i)^\circ))$ can be written in the form a^b for positive integers a, b with a squarefree, find $a + b$.

Proposed by Deyuan Li and Andrew Milas

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12 $ABCD$ is a regular tetrahedron with side length 1. Points X, Y, and Z, distinct from A, B, and C, respectively, are drawn such that $BCDX$, $ACDY$, and $ABDZ$ are also regular tetrahedra. If the volume of the polyhedron with faces ABC, XYZ, BXC, XCY, CYA, YAZ, AZB, and

ZBX can be written as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with $\gcd(a, c) = 1$ and b squarefree, find $a + b + c$.

Proposed by Jason Wang

– Tiebreaker

1 Let a, b, c be the roots of the polynomial $x^3 - 20x^2 + 22$. Find

$$\frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2}.$$

Proposed by Deyuan Li and Andrew Milas

2 In any finite grid of squares, some shaded and some not, for each unshaded square, record the number of shaded squares horizontally or vertically adjacent to it; this grid's *score* is the sum of all numbers recorded this way. Deyuan shades each square in a blank $n \times n$ grid with probability k; he notices that the expected value of the score of the resulting grid is equal to k, too! Given that $k > 0.9999$, find the minimum possible value of n.

Proposed by Andrew Wu

3 Find the sum of all x from 2 to 1000 inclusive such that

$$\prod_{n=2}^{x} \log_{n^n}(n+1)^{n+2}$$

is an integer.

Proposed by Deyuan Li and Andrew Milas

4 Let triangle ABC with incenter I and circumcircle Γ satisfy $AB = 6\sqrt{3}$, $BC = 14$, and $CA = 22$. Construct points P and Q on rays BA and CA such that $BP = CQ = 14$. Lines PI and QI meet the tangents from B and C to Γ, respectively, at points X and Y. If XY can be expressed as $a\sqrt{b} - c$ for positive integers a, b, c with c squarefree, find $a + b + c$.

Proposed by Andrew Wu

– Mathathon Round

– Round 1

p1. Ben the bear has an algorithm he runs on positive integers- each second, if the integer is even, he divides it by 2, and if the integer is odd, he adds 1. The algorithm terminates after he reaches 1. What is the least positive integer n such that Ben's algorithm performed on n will terminate after seven seconds? (For example, if Ben performed his algorithm on 3, the algorithm would terminate after 3 seconds: $3 \to 4 \to 2 \to 1$.)

p2. Suppose that a rectangle R has length p and width q, for prime integers p and q. Rectangle S has length $p+1$ and width $q+1$. The absolute difference in area between S and R is 21. Find the sum of all possible values of p.

p3. Owen the origamian takes a rectangular 12×16 sheet of paper and folds it in half, along the diagonal, to form a shape. Find the area of this shape.

Round 2

p4. How many subsets of the set $\{G, O, Y, A, L, E\}$ contain the same number of consonants as vowels? (Assume that Y is a consonant and not a vowel.)

p5. Suppose that trapezoid $ABCD$ satisfies $AB = BC = 5$, $CD = 12$, and $\angle ABC = \angle BCD = 90^o$. Let AC and BD intersect at E. The area of triangle BEC can be expressed as $\frac{a}{b}$, for positive integers a and b with $gcd(a, b) = 1$. Find $a + b$.

p6. Find the largest integer n for which $\frac{101^n+103^n}{101^{n-1}+103^{n-1}}$ is an integer.

Round 3

p7. For each positive integer n between 1 and 1000 (inclusive), Ben writes down a list of n's factors, and then computes the median of that list. He notices that for some n, that median is actually a factor of n. Find the largest n for which this is true.

p8. (voided) Suppose triangle ABC has $AB = 9$, $BC = 10$, and $CA = 17$. Let x be the maximal possible area of a rectangle inscribed in ABC, such that two of its vertices lie on one side and the other two vertices lie on the other two sides, respectively. There exist three rectangles R_1, R_2, and R_3 such that each has an area of x. Find the area of the smallest region containing the set of points that lie in at least two of the rectangles R_1, R_2, and R_3.

p9. Let a, b, and c be the three smallest distinct positive values of θ satisfying

$$\cos\theta + \cos 3\theta + ... + \cos 2021\theta = \sin\theta + \sin 3\theta + ... + \sin 2021\theta.$$

What is $\frac{4044}{\pi}(a + b + c)$?

Problem 8 is voided.

PS. You should use hide for answers.Rounds 4-5 have been posted here (https://artofproblemsolving.com/community/c4h3131422p28368457) and 6-7 here (https://artofproblemsolving.com/community/c4h3131434p28368604). Collected here (https://artofproblemsolving.com/community/c5h2760506p24

– Round 4

p10. How many divisors of 10^{11} have at least half as many divisors that 10^{11} has?

p11. Let $f(x,y) = \frac{x}{y} + \frac{y}{x}$ and $g(x,y) = \frac{x}{y} - \frac{y}{x}$. Then, if $\underbrace{f(f(...f(f(}_{2021fs}f(f(1,2),g(2,1)),2),2)...,2),2)$ can be expressed in the form $a + \frac{b}{c}$, where a, b,c are nonnegative integers such that $b < c$ and $gcd(b,c) = 1$, find $a + b + \lceil(\log_2(\log_2 c)\rceil$

p12. Let ABC be an equilateral triangle, and letDEF be an equilateral triangle such that D, E, and F lie on AB, BC, and CA, respectively. Suppose that AD and BD are positive integers, and that $\frac{[DEF]}{[ABC]} = \frac{97}{196}$. The circumcircle of triangle DEF meets AB, BC, and CA again at G, H, and I, respectively. Find the side length of an equilateral triangle that has the same area as the hexagon with vertices D, E, F, G, H, and I.

Round 5

p13. Point X is on line segment AB such that $AX = \frac{2}{5}$ and $XB = \frac{5}{2}$. Circle Ω has diameter AB and circle ω has diameter XB. A ray perpendicular to AB begins at X and intersects Ω at a point Y. Let Z be a point on ω such that $\angle YZX = 90^o$. If the area of triangle XYZ can be expressed as $\frac{a}{b}$ for positive integers a, b with $gcd(a,b) = 1$, find $a + b$.

p14. Andrew, Ben, and Clayton are discussing four different songs; for each song, each person either likes or dislikes that song, and each person likes at least one song and dislikes at least one song. As it turns out, Andrew and Ben don't like any of the same songs, but Clayton likes at least one song that Andrew likes and at least one song that Ben likes! How many possible ways could this have happened?

p15. Let triangle ABC with circumcircle Ω satisfy $AB = 39$, $BC = 40$, and $CA = 25$. Let P be a point on arc BC not containing A, and let Q and R be the reflections of P in AB and AC, respectively. Let AQ and AR meet Ω again at S and T, respectively. Given that the reflection of

QR over BC is tangent to Ω , ST can be expressed as $\frac{a}{b}$ for positive integers a, b with $gcd(a, b) = 1$. Find $a + b$.

PS. You should use hide for answers. Rounds 1-3 have been posted here (https://artofproblemsolving.com/community/c4h3131401p28368159) and 6-7 here (https://artofproblemsolving.com/community/c4h3131434p28368604),Collected here (https://artofproblemsolving.com/community/c5h2760506p241

– Round 6

p16. Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. There exist two possible points X on CA such that if Y and Z are the feet of the perpendiculars from X to AB and BC, respectively, then the area of triangle XYZ is 1. If the distance between those two possible points can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c with b squarefree and $gcd(a, c) = 1$, then find $a + b + c$.

p17. Let $f(n)$ be the number of orderings of $1, 2, ..., n$ such that each number is as most twice the number preceding it. Find the number of integers k between 1 and 50, inclusive, such that $f(k)$ is a perfect square.

p18. Suppose that f is a function on the positive integers such that $f(p) = p$ for any prime p, and that $f(xy) = f(x) + f(y)$ for any positive integers x and y. Define $g(n) = \sum_{k|n} f(k)$; that is, $g(n)$ is the sum of all $f(k)$ such that k is a factor of n. For example, $g(6) = f(1) + 1(2) + f(3) + f(6)$. Find the sum of all composite n between 50 and 100, inclusive, such that $g(n) = n$.

Round 7

p19. AJ is standing in the center of an equilateral triangle with vertices labelled A, B, and C. They begin by moving to one of the vertices and recording its label; afterwards, each minute, they move to a different vertex and record its label. Suppose that they record 21 labels in total, including the initial one. Find the number of distinct possible ordered triples (a, b, c), where a is the number of A's they recorded, b is the number of B's they recorded, and c is the number of C's they recorded.

p20. Let $S = \sum_{n=1}^{\infty}(1 - \{(2 + \sqrt{3})^n\})$, where $\{x\} = x - \lfloor x \rfloor$, the fractional part of x. If $S = \frac{\sqrt{a}-b}{c}$ for positive integers a, b, c with a squarefree, find $a + b + c$.

p21. Misaka likes coloring. For each square of a 1×8 grid, she flips a fair coin and colors in the square if it lands on heads. Afterwards, Misaka places as many 1×2 dominos on the grid as possible such that both parts of each domino lie on uncolored squares and no dominos overlap. Given that the expected number of dominos that she places can be written as $\frac{a}{b}$, for positive integers a and b with $gcd(a, b) = 1$, find $a + b$.

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– Mixer Round

Mixer Round **p1.** Prair takes some set S of positive integers, and for each pair of integers she computes the positive difference between them. Listing down all the numbers she computed, she notices that every integer from 1 to 10 is on her list! What is the smallest possible value of $|S|$, the number of elements in her set S?

p2. Jake has 2021 balls that he wants to separate into some number of bags, such that if he wants any number of balls, he can just pick up some bags and take all the balls out of them. What is the least number of bags Jake needs?

p3. Claire has stolen Cat's scooter once again! She is currently at (0; 0) in the coordinate plane, and wants to ride to $(2, 2)$, but she doesn't know how to get there. So each second, she rides one unit in the positive x or y-direction, each with probability $\frac{1}{2}$. If the probability that she makes it to $(2, 2)$ during her ride can be expressed as $\frac{a}{b}$ for positive integers a, b with $gcd(a, b) = 1$, then find $a + b$.

p4. Triangle ABC with $AB = BC = 6$ and $\angle ABC = 120^o$ is rotated about A, and suppose that the images of points B and C under this rotation are B' and C', respectively. Suppose that A, B' and C are collinear in that order. If the area of triangle $B'CC'$ can be expressed as $a - b\sqrt{c}$ for positive integers a, b, c with csquarefree, find $a + b + c$.

p5. Find the sum of all possible values of $a + b + c + d$ if a, b, c,d are positive integers satisfying

$$ab + cd = 100,$$

$$ac + bd = 500.$$

p6. Alex lives in Chutes and Ladders land, which is set in the coordinate plane. Each step they take brings them one unit to the right or one unit up. However, there's a chute-ladder between points $(1, 2)$ and $(2, 0)$ and a chute-ladder between points $(1, 3)$ and $(4, 0)$, whenever Alex visits an endpoint on a chute-ladder, they immediately appear at the other endpoint of that chute-ladder! How many ways are there for Alex to go from $(0, 0)$ to $(4, 4)$?

p7. There are 8 identical cubes that each belong to 8 different people. Each person randomly picks a cube. The probability that exactly 3 people picked their own cube can be written as $\frac{a}{b}$, where a and b are positive integers with $gcd(a, b) = 1$. Find $a + b$.

p8. Suppose that $p(R) = Rx^2 + 4x$ for all R. There exist finitely many integer values of R such that $p(R)$ intersects the graph of $x^3 + 2021x^2 + 2x + 1$ at some point (j, k) for integers j and k. Find the sum of all possible values of R.

p9. Let a, b, c be the roots of the polynomial $x^3 - 20x^2 + 22$. Find $\frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2}$.

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p12. Let triangle ABC with incenter I and circumcircle Γ satisfy $AB = 6\sqrt{3}$, $BC = 14$, and $CA = 22$. Construct points P and Q on rays BA and CA such that $BP = CQ = 14$. Lines PI and QI meet the tangents from B and C to Γ, respectively, at points X and Y . If XY can be expressed as $a\sqrt{b} - c$ for positive integers a, b, c with c squarefree, find $a + b + c$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).