www.artofproblemsolving.com/community/c2550819
by parmenides51, wu2481632

– Individual Round

1 Given that $2x+7y=3$, find $2^{6x+21y-4}$.

Proposed by Deyuan Li

2 A box of strawberries, containing 12 strawberries total, costs $\$2$. A box of blueberries, containing 48 blueberries total, costs $\$3$. Suppose that for $\$12$, Sareen can either buy m strawberries total or n blueberries total. Find $n-m$.

Proposed by Andrew Wu

3 Suppose that $a_1=1$, $a_2=2$, and for any $n\geq 3$, $a_n=a_1+a_2+\cdots+a_{n-1}$. Find $\frac{a_{2021}}{a_{2020}}$.

Proposed by Andrew Wu

4 Cat and Claire are having a conversation about Cat's favorite number.

Cat says, "My favorite number is a two-digit positive integer that is the product of three distinct prime numbers!"

Claire says, "I don't know your favorite number yet, but I do know that among four of the numbers that might be your favorite number, you could start with any one of them, add a second, subtract a third, and get the fourth!"

Cat says, "That's cool! My favorite number is not among those four numbers, though."

Claire says, "Now I know your favorite number!"

What is Cat's favorite number?

Proposed by Andrew Wu and Andrew Milas

5 Let $ATHEM$ be a convex pentagon with $AT=14$, $TH=MA=20$, $HE=EM=15$, and $\angle THE=\angle EMA=90^\circ$. Find the area of $ATHEM$.

Proposed by Andrew Wu

6 Kara rolls a six-sided die six times, and notices that the results satisfy the following conditions:

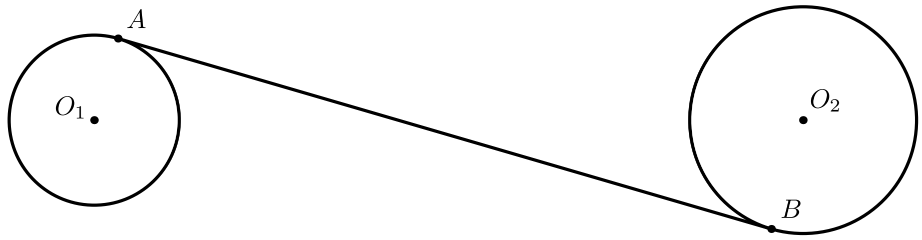
- She rolled a 6 exactly three times;
- The product of her first three rolls is the same as the product of her last three rolls.

How many distinct sequences of six rolls could Kara have rolled?

Proposed by Andrew Wu

---

7 Suppose two circles Ω_1 and Ω_2 with centers O_1 and O_2 have radii 3 and 4, respectively. Suppose that points A and B lie on circles Ω_1 and Ω_2, respectively, such that segments AB and O_1O_2 intersect and that AB is tangent to Ω_1 and Ω_2. If $O_1O_2 = 25$, find the area of quadrilateral O_1AO_2B.



Proposed by Deyuan Li and Andrew Milas

---

8 Let A and B be digits between 0 and 9, and suppose that the product of the two-digit numbers $\overline{AB}$ and $\overline{BA}$ is equal to k. Given that $k+1$ is a multiple of 101, find k.

Proposed by Andrew Wu

---

9 Ali defines a *pronunciation* of any sequence of English letters to be a partition of those letters into substrings such that each substring contains at least one vowel. For example, A | THEN | A, ATH | E | NA, ATHENA, and AT | HEN | A are all pronunciations of the sequence ATHENA. How many distinct pronunciations does YALEMATHCOMP have? (Y is not a vowel.)

Proposed by Andrew Wu, with significant inspiration from ali cy

---

10 Suppose that $a_1, a_2, a_3, \ldots$ is an infinite geometric sequence such that for all $i \geq 1$, a_i is a positive integer. Suppose furthermore that $a_{20} + a_{21} = 20^{21}$. If the minimum possible value of a_1 can be expressed as $2^a 5^b$ for positive integers a and b, find $a + b$.

Proposed by Andrew Wu

---

11 A right rectangular prism has integer side lengths a, b, and c. If $\text{lcm}(a, b) = 72$, $\text{lcm}(a, c) = 24$, and $\text{lcm}(b, c) = 18$, what is the sum of the minimum and maximum possible volumes of the prism?

Proposed by Deyuan Li and Andrew Milas

---

12 Let Γ_1 and Γ_2 be externally tangent circles with radii lengths 2 and 6, respectively, and suppose that they are tangent to and lie on the same side of line ℓ. Points A and B are selected on ℓ such that Γ_1 and Γ_2 are internally tangent to the circle with diameter AB. If $AB = a+b\sqrt{c}$ for positive integers a, b, c with c squarefree, then find $a+b+c$.

Proposed by Andrew Wu, Deyuan Li, and Andrew Milas

Tiebreaker p1. In their class Introduction to Ladders at Greendale Community College, Jan takes four tests. They realize that their test scores in chronological order form a strictly increasing arithmetic progression with integer terms, and that the average of those scores is an integer greater than or equal to 94. How many possible combinations of test scores could they have had? (Test scores at Greendale range between 0 and 100, inclusive.)

p2. Suppose that A and B are digits between 1 and 9 such that

$$0.\overline{ABABAB...} + B \cdot (0.\overline{AAA...}) = A \cdot (0.\overline{B1B1B1...}) + 1$$

Find the sum of all possible values of $10A + B$.

p3. Let ABC be an isosceles right triangle with $m\angle ABC = 90^o$. Let D and E lie on segments $\overline{AC}$ and $\overline{BC}$, respectively, such that triangles $\vartriangle ADB$ and $\vartriangle CDE$ are similar and $DE = EB$. If $\frac{AC}{AD} = 1 + \frac{\sqrt{a}}{b}$ with a, b positive integers and a squarefree, then find $a + b$.

p4. Five bowling pins $P_1, P_2, ..., P_5$ are lined up in a row. Each turn, Jemma picks a pin at random from the standing pins, and throws a bowling ball at that pin; that pin and each pin directly adjacent to it are knocked down. If the expected value of the number of turns Jemma will take to knock down all the pins is $\frac{a}{b}$ where a and b are relatively prime, find $a + b$. (Pins P_i and P_j are adjacent if and only if $|i - j| = 1$.)

p5. How many terms in the expansion of

$$(1 + x + x^2 + x^3 + ... + x^{2021})(1 + x^2 + x^4 + x^6 + ... + x^{4042})$$

have coeffcients equal to 1011?

p6. Suppose $f(x)$ is a monic quadratic polynomial with distinct nonzero roots p and q, and suppose $g(x)$ is a monic quadratic polynomial with roots $p + \frac{1}{q}$ and $q + \frac{1}{p}$. If we are given that $g(-1) = 1$ and $f(0) \ne -1$, then there exists some real number r that must be a root of $f(x)$. Find r.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

–	Mathathon
R1	1. If $5x + 3y - z = 4$, $x = y$, and $z = 4$, find $x + y + z$. 2. How many ways are there to pick three distinct vertices of a regular hexagon such that the triangle with those three points as its vertices shares exactly one side with the hexagon? 3. Sirena picks five distinct positive primes, $p_1 < p_2 < p_3 < p_4 < p_5$, and finds that they sum to 192. If the product $p_1p_2p_3p_4p_5$ is as large as possible, what is $p_1 - p_2 + p_3 - p_4 + p_5$?
R2	4. Suppose that $\overline{A2021B}$ is a six-digit integer divisible by 9. Find the maximum possible value of $A \cdot B$. 5. In an arbitrary triangle, two distinct segments are drawn from each vertex to the opposite side. What is the minimum possible number of intersection points between these segments? 6. Suppose that a and b are positive integers such that $\frac{a}{b-20}$ and $\frac{b+21}{a}$ are positive integers. Find the maximum possible value of $a + b$.
R3	7. Peggy picks three positive integers between 1 and 25, inclusive, and tells us the following information about those numbers: - Exactly one of them is a multiple of 2; - Exactly one of them is a multiple of 3; - Exactly one of them is a multiple of 5; - Exactly one of them is a multiple of 7; - Exactly one of them is a multiple of 11. What is the maximum possible sum of the integers that Peggy picked? 8. What is the largest positive integer k such that 2^k divides $2^{4^8} + 8^{2^4} + 4^{8^2}$? 9. Find the smallest integer n such that n is the sum of 7 consecutive positive integers and the sum of 12 consecutive positive integers.
R4	10. Prair picks a three-digit palindrome n at random. If the probability that $2n$ is also a palindrome can be expressed as $\frac{p}{q}$ in simplest terms, find $p + q$. (A palindrome is a number that reads the same forwards as backwards; for example, 161 and 2992 are palindromes, but 342 is not.) 11. If two distinct integers are picked randomly between 1 and 50 inclusive, the probability that their sum is divisible by 7 can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$. 12. Ali is playing a game involving rolling standard, fair six-sided dice. She calls two consecutive die rolls such that the first is less than the second a ”rocket.” If, however, she ever rolls two

consecutive die rolls such that the second is less than the first, the game stops. If the probability that Ali gets five rockets is $\frac{p}{q}$, where p and q are relatively prime positive integers, find $p+q$.

R5 13. The triangle with vertices $(0,0)$, (a,b), and $(a,-b)$ has area 10. Find the sum of all possible positive integer values of a, given that b is a positive integer.

14. Elsa is venturing into the unknown. She stands on $(0,0)$ in the coordinate plane, and each second, she moves to one of the four lattice points nearest her, chosen at random and with equal probability. If she ever moves to a lattice point she has stood on before, she has ventured back into the known, and thus stops venturing into the unknown from then on. After four seconds have passed, the probability that Elsa is still venturing into the unknown can be expressed as $\frac{a}{b}$ in simplest terms. Find $a+b$.

(A lattice point is a point with integer coordinates.)

15. Let $ABCD$ be a square with side length 4. Points X,Y, and Z, distinct from points A,B,C, and D, are selected on sides AD, AB, and CD, respectively, such that $XY = 3, XZ = 4$, and $\angle YXZ = 90^\circ$. If $AX = \frac{a}{b}$ in simplest terms, then find $a+b$.

R6 16. Suppose trapezoid $JANE$ is inscribed in a circle of radius 25 such that the center of the circle lies inside the trapezoid. If the two bases of $JANE$ have side lengths 14 and 30 and the average of the lengths of the two legs is $\sqrt{m}$, what is m?

17. What is the radius of the circle tangent to the x-axis, the line $y = \sqrt{3}x$, and the circle $(x - 10\sqrt{3})^2 + (y-10)^2 = 25$?

18. Find the smallest positive integer n such that $3n^3 - 9n^2 + 5n - 15$ is divisible by 121 but not 2.

Mixer Round **p1.** Find the number of ordered triples (a,b,c) satisfying • a,b,c are are single-digit positive integers, and • $a \cdot b + c = a + b \cdot c$.

p2. In their class Introduction to Ladders at Greendale Community College, Jan takes four tests. They realize that their test scores in chronological order form an increasing arithmetic progression with integer terms, and that the average of those scores is an integer greater than or equal to 94. How many possible combinations of test scores could they have had? (Test scores at Greendale range between 0 and 100, inclusive.)

p3. Suppose that $a + \frac{1}{b} = 2$ and $b + \frac{1}{a} = 3$. If $\frac{a}{b} + \frac{b}{a}$ can be expressed as $\frac{p}{q}$ in simplest terms, find $p+q$.

p4. Suppose that A and B are digits between 1 and 9 such that

$$0.\overline{ABABAB...} + B \cdot (0.\overline{AAA...}) = A \cdot (0.\overline{B_1B_1B_1...}) + 1$$

Find the sum of all possible values of $10A + B$.

p5. Let ABC be an isosceles right triangle with $m\angle ABC = 90^o$. Let D and E lie on segments AC and BC, respectively, such that triangles $\triangle ADB$ and $\triangle CDE$ are similar and $DE = EB$. If $\frac{AC}{AD} = 1 + \frac{\sqrt{a}}{b}$ with a, b positive integers and a squarefree, then find $a + b$.

p6. Five bowling pins P_1, P_2,..., P_5 are lined up in a row. Each turn, Jemma picks a pin at random from the standing pins, and throws a bowling ball at that pin; that pin and each pin directly adjacent to it are knocked down. If the expected value of the number of turns Jemma will take to knock down all the pins is a b where a and b are relatively prime, find $a + b$. (Pins P_i and P_j are adjacent if and only if $|i - j| = 1$.)

p7. Let triangle ABC have side lengths $AB = 10$, $BC = 24$, and $AC = 26$. Let I be the incenter of ABC. If the maximum possible distance between I and a point on the circumcircle of ABC can be expressed as $a + \sqrt{b}$ for integers a and b with b squarefree, find $a + b$.
(The incenter of any triangle XYZ is the intersection of the angle bisectors of $\angle YXZ$, $\angle XZY$, and $\angle ZYX$.)

p8. How many terms in the expansion of

$$(1 + x + x^2 + x^3 + ... + x^{2021})(1 + x^2 + x^4 + x^6 + ... + x^{4042})$$

have coefficients equal to 1011?

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).