China National Olympiad 2016

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Day 1 December 16th

1 Let $a_1, a_2, \cdots, a_{31}; b_1, b_2, \cdots, b_{31}$ be positive integers such that $a_1 < a_2 < \cdots < a_{31} \leq 2015$, $b_1 < b_2 < \cdots < b_{31} \leq 2015$ and $a_1 + a_2 + \cdots + a_{31} = b_1 + b_2 + \cdots + b_{31}$.
Find the maximum value of $S = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{31} - b_{31}|$.

2 In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C. Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively.
Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

3 Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

1) There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod p$ for all $1 \leq i \leq p$

2) For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod p$$

where indices are taken $\pmod p$

Day 2 December 17th

4 Let $n \geq 2$ be a positive integer and define k to be the number of primes $\leq n$. Let A be a subset of $S = \{2, ..., n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B such that $|B| = k$, $A \subseteq B \subseteq S$ and no two elements in B divide each other.

5 Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (Vertices maybe ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

6 Let G be a complete directed graph with 100 vertices such that for any two vertices x, y one can find a directed path from x to y.

a) Show that for any such G, one can find a m such that for any two vertices x, y one can find a directed path of length m from x to y (Vertices can be repeated in the path)

b) For any graph G with the properties above, define $m(G)$ to be smallest possible m as defined in part a). Find the minimim value of $m(G)$ over all such possible G's.
