2015 Japan MO Finals

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1 Find all positive integers n such that $\frac{10^n}{n^3+n^2+n+1}$ is an integer.

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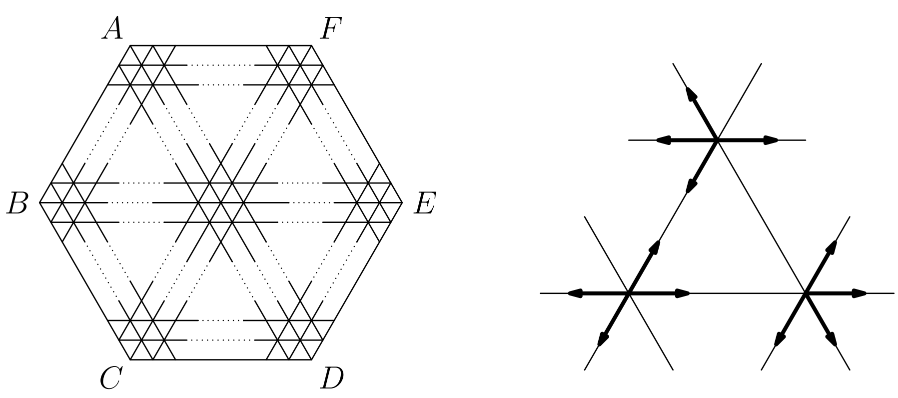
2 Let n be an integer greater than or equal to 2. There is a regular hexagon $ABCDEF$ with side length n, which is divided into equilateral triangles with side length 1 as shown in the left figure. We simply call the vertices of an equilateral triangle as vertex.
A piece is placed in the center of a regular hexagon $ABCDEF$. For each of the vertices P inside $ABCDEF$ (not including the perimeter), an arrow is drawn toward 4 vertices four of the six vertices connected to P by an edge of length 1. When a piece is placed on the vertex P, this piece can be moved to any of the four vertices which the arrow is drawn. Regarding side PQ, even if a piece can be moved from vertex P to vertex Q, it is not necessarily possible to move a piece from vertex Q to vertex P.
Then, show that there exists an integer k such that it can be moved the piece to vertices on the perimeter of the regular hexagon $ABCDEF$ in k moves regardless of how the arrows are drawn and find the smallest possible value of k.



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3 A sequence $\{a_n\}_{n\geq 1}$ of positive integers is called *ascending* if a_n satisfies $a_n < a_{n+1}$ and $a_{2n} = 2a_n$.

(1) Let $\{a_n\}$be *ascending*. If p is a prime greater than a_1, then prove that there exists a multiple of p in the sequence.

(2) Let p be an odd prime. Prove that there exists a sequence $\{a_n\}$ which is *ascending* and has no multiple of p.

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4 Scalene triangle ABC has circumcircle Γ and incenter I. The incircle of triangle ABC touches side AB, AC at D, E respectively. Circumcircle of triangle BEI intersects Γ again at P distinct from B, circumcircle of triangle CDI intersects Γ again at Q distinct from C. Prove that the 4 points D, E, P, Q are concyclic.

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5 Let a be a fixed positive integer. For a given positive integer n, consider the following assertion.

In an infinite two-dimensional grid of squares, n different cells are colored black. Let K denote the number of a by a squares in the grid containing exactly a black cells. Then over all possible choices of the n black cells, the maximum possible K is $a(n+1-a)$.

Prove that there exists a positive integer N such that for all $n \geq N$, this assertion is true.

(link is http://www.imojp.org/challenge/old/jmo25mq.html for anyone who wants to correct my translation)
