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1 Let p be an odd prime number. For positive integer k satisfying $1 \leq k \leq p-1$, the number of divisors of $kp+1$ between k and p exclusive is a_k. Find the value of $a_1 + a_2 + \ldots + a_{p-1}$.

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2 Let $ABCD$ be a concyclic quadrilateral such that $AB : AD = CD : CB$. The line AD intersects the line BC at X, and the line AB intersects the line CD at Y. Let $E,\ F,\ G$ and H are the midpoints of the edges $AB,\ BC,\ CD$ and DA respectively. The bisector of angle AXB intersects the segment EG at S, and that of angle AYD intersects the segment FH at T. Prove that the lines ST and BD are pararell.

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3 Let n be a positive integer. In JMO kingdom there are 2^n citizens and a king. In terms of currency, the kingdom uses paper bills with value \$$2^n$ and coins with value \$$2^a (a = 0, 1 \ldots, n-1)$. Every citizen has infinitely many paper bills. Let the total number of coins in the kingdom be S. One fine day, the king decided to implement a policy which is to be carried out every night:
- Each citizen must decide on a finite amount of money based on the coins that he currently has, and he must pass that amount to either another citizen or the king;
-Each citizen must pass exactly \$1 more than the amount he received from other citizens.

Find the minimum value of S such that the king will be able to collect money every night eternally.

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4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x) - x) = f(x)f(y) + 2x$$

for all $x,\ y \in \mathbb{R}$.

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5 m, n are positive integers such that $m \geq 2$, $n < \frac{3}{2}(m-1)$. In a country there are m cities and n roads, each road connect two different cities, and there can be multiple roads between two cities. Prove that there exist a way to separate the cities into two groups α and β, where all roads connecting a city in α to a city in β is converted to a highway, and satisfies the following conditions:
-Both groups have at least one city, and
-for each city, the number of highways coming out from that city does not exceed 1.
