NIMO Problems 2016

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by MSTang, ABCDE, djmathman, Binomial-theorem

Day 20 September 23, 2015

1 Three fair six-sided dice are labeled with the numbers $\{1, 2, 3, 4, 5, 6\}$, $\{1, 2, 3, 4, 5, 6\}$, and $\{1, 2, 3, 7, 8, 9\}$, respectively. All three dice are rolled. The probability that at least two of the dice have the same value is m/n, where m, n are relatively prime positive integers. Find $100m + n$.

Proposed by Michael Tang

2 Define the *hotel elevator cubic* as the unique cubic polynomial P for which $P(11) = 11$, $P(12) = 12$, $P(13) = 14$, $P(14) = 15$. What is $P(15)$?

Proposed by Evan Chen

3 How many positive integers divide at least two of the numbers 120, 144, and 180?

Proposed by Evan Chen

4 Let $f(n) = \frac{n}{3}$ if n is divisible by 3 and $f(n) = 4n - 10$ otherwise. Find the sum of all positive integers c such that $f^5(c) = 2$. (Here $f^5(x)$ means $f(f(f(f(f(x)))))$.)

Proposed by Justin Stevens

5 For positive integers n, let $s(n)$ be the sum of the digits of n. Over all four-digit positive integers n, which value of n maximizes the ratio $\frac{s(n)}{n}$?

Proposed by Michael Tang

6 Let ABC be a triangle with $AB = 20$, $AC = 34$, and $BC = 42$. Let ω_1 and ω_2 be the semicircles with diameters $\overline{AB}$ and $\overline{AC}$ erected outwards of $\triangle ABC$ and denote by ℓ the common external tangent to ω_1 and ω_2. The line through A perpendicular to $\overline{BC}$ intersects ℓ at X and BC at Y. The length of $\overline{XY}$ can be written in the form $m + \sqrt{n}$ where m and n are positive integers. Find $100m + n$.

Proposed by David Altizio

7 Suppose a, b, c, and d are positive real numbers which satisfy the system of equations

$$\begin{aligned} a^2+b^2+c^2+d^2 &= 762, \\ ab+cd &= 260, \\ ac+bd &= 365, \\ ad+bc &= 244. \end{aligned}$$

Compute $abcd$.

Proposed by Michael Tang

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8 Justin the robot is on a mission to rescue abandoned treasure from a minefield. To do this, he must travel from the point $(0,0,0)$ to $(4,4,4)$ in three-dimensional space, only taking one-unit steps in the positive $x, y,$ or z-directions. However, the evil David anticipated Justin's arrival, and so he has surreptitiously placed a mine at the point $(2,2,2)$. If at any point Justin is at most one unit away from this mine (in any direction), the mine detects his presence and explodes, thwarting Justin.

How many paths can Justin take to reach his destination safely?

Proposed by Justin Stevens

Day 21 October 16, 2015

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1 In triangle ABC, $AB=13$, $BC=14$, and $CA=15$. A circle of radius r passes through point A and is tangent to line BC at C. If $r=m/n$, where m and n are relatively prime positive integers, find $100m+n$.

Proposed by Michael Tang

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2 In the *Fragmented Game of Spoons*, eight players sit in a row, each with a hand of four cards. Each round, the first player in the row selects the top card from the stack of unplayed cards and either passes it to the second player, which occurs with probability $\frac{1}{2}$, or swaps it with one of the four cards in his hand, each card having an equal chance of being chosen, and passes the new card to the second player. The second player then takes the card from the first player and chooses a card to pass to the third player in the same way. Play continues until the eighth player is passed a card, at which point the card he chooses to pass is removed from the game and the next round begins. To win, a player must hold four cards of the same number, one of each suit.

During a game, David is the eighth player in the row and needs an Ace of Clubs to win. At the start of the round, the dealer picks up a Ace of Clubs from the deck. Suppose that Justin, the fifth player, also has a Ace of Clubs, and that all other Ace of Clubs cards have been removed. The probability that David is passed an Ace of Clubs during the round is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m,n)=1$. Find $100m+n$.

Proposed by David Altizio

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3 Find the sum of all positive integers n such that exactly 2% of the numbers in the set $\{1, 2, \dots, n\}$ are perfect squares.

Proposed by Michael Tang

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4 Let $f(x, y)$ be a function defined for all pairs of nonnegative integers (x, y), such that $f(0, k) = f(k, 0) = 2^k$ and

$$f(a, b) + f(a+1, b+1) = f(a+1, b) + f(a, b+1)$$

for all nonnegative integers a, b. Determine the number of positive integers $n \leq 2016$ for which there exist two nonnegative integers a, b such that $f(a, b) = n$.

Proposed by Michael Ren

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5 Find the constant k such that the sum of all $x \geq 0$ satisfying $\sqrt{x}(x+12) = 17x - k$ is 256.

Proposed by Michael Tang

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6 Let $ABCD$ be an isosceles trapezoid with $AD \parallel BC$ and $BC > AD$ such that the distance between the incenters of $\triangle ABC$ and $\triangle DBC$ is 16. If the perimeters of $ABCD$ and ABC are 120 and 114 respectively, then the area of $ABCD$ can be written as $m\sqrt{n}$, where m and n are positive integers with n not divisible by the square of any prime. Find $100m + n$.

Proposed by David Altizio and Evan Chen

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7 Given two positive integers m and n, we say that $m \mid\mid n$ if $m \mid n$ and $\gcd(m,\, n/m) = 1$. Compute the smallest integer greater than

$$\sum_{d \mid 2016} \sum_{m \mid\mid d} \frac{1}{m}.$$

Proposed by Michael Ren

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8 Triangle ABC has $AB = 25$, $AC = 29$, and $BC = 36$. Additionally, Ω and ω are the circumcircle and incircle of $\triangle ABC$. Point D is situated on Ω such that AD is a diameter of Ω, and line AD intersects ω in two distinct points X and Y. Compute XY^2.

Proposed by David Altizio

Day 22 December 8, 2015

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1 Find the value of $\lfloor 1 \rfloor + \lfloor 1.7 \rfloor + \lfloor 2.4 \rfloor + \lfloor 3.1 \rfloor + \cdots + \lfloor 99 \rfloor$.

Proposed by Jack Cornish

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2 Sitting at a desk, Alice writes a nonnegative integer N on a piece of paper, with $N \leq 10^{10}$. Interestingly, Celia, sitting opposite Alice at the desk, is able to properly read the number upside-down and gets the same number N, without any leading zeros. (Note that the digits 2, 3, 4, 5, and 7 will not be read properly when turned upside-down.) Find the number of possible values of N.

Proposed by Yannick Yao

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3 Right triangle ABC has hypotenuse $AB = 26$, and the inscribed circle of ABC has radius 5. The largest possible value of BC can be expressed as $m+\sqrt{n}$, where m and n are both positive integers. Find $100m + n$.

Proposed by Jason Xia

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4 In rhombus $ABCD$, let M be the midpoint of AB and N be the midpoint of AD. If $CN = 7$ and $DM = 24$, compute AB^2.

Proposed by Andy Liu

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5 In a chemistry experiment, a tube contains 100 particles, 68 on the right and 32 on the left. Each second, if there are a particles on the left side of the tube, some number n of these particles move to the right side, where $n \in \{0, 1, \dots, a\}$ is chosen uniformly at random. In a similar manner, some number of the particles from the right side of the tube move to the left, at the same time. The experiment ends at the moment when all particles are on the same side of the tube. The probability that all particles end on the left side is $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Proposed by Alvin Zou

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6 Consider a sequence a_0, a_1, $\dots$, a_9 of distinct positive integers such that $a_0 = 1$, $a_i < 512$ for all i, and for every $1 \leq k \leq 9$ there exists $0 \leq m \leq k-1$ such that

$$(a_k - 2a_m)(a_k - 2a_m - 1) = 0.$$

Let N be the number of these sequences. Find the remainder when N is divided by 1000.

Based on a proposal by Gyumin Roh

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7 Determine the number of odd integers $1 \leq n \leq 100$ with the property that

$$\sum_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} \cos\left(\frac{2\pi k}{n}\right) = 1 \quad \text{and} \quad \sum_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} \sin\left(\frac{2\pi k}{n}\right) = 0.$$

Based on a proposal by Mayank Pandey

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8 Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ be real numbers which satisfy

$$S_3 = S_{11} = 1, \quad S_7 = S_{15} = -1, \quad \text{and} \quad S_5 = S_9 = S_{13} = 0, \quad \text{where} \quad S_n = \sum_{\substack{1 \le i < j \le 8 \\ i+j=n}} a_i a_j.$$

(For example, $S_5 = a_1a_4 + a_2a_3$.)
Assuming $|a_1| = |a_2| = 1$, the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_8^2$ can be written as $a + \sqrt{b}$ for integers a and b. Compute $a + b$.

Based on a proposal by Nathan Soedjak

Day 23 January 7, 2016

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1 Let m be a positive integer less than 2015. Suppose that the remainder when 2015 is divided by m is n. Compute the largest possible value of n.

Proposed by Michael Ren

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2 For real numbers x and y, define

$$\nabla(x, y) = x - \frac{1}{y}.$$

If

$$\underbrace{\nabla(2, \nabla(2, \nabla(2, \ldots \nabla(2, \nabla(2, 2)) \ldots)))}_{2016 \ \nabla\text{s}} = \frac{m}{n}$$

for relatively prime positive integers m, n, compute $100m + n$.

Proposed by David Altizio

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3 Convex pentagon $ABCDE$ satisfies $AB \parallel DE$, $BE \parallel CD$, $BC \parallel AE$, $AB = 30$, $BC = 18$, $CD = 17$, and $DE = 20$. Find its area.

Proposed by Michael Tang

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4 A fair 100-sided die is rolled twice, giving the numbers a and b in that order. If the probability that $a^2 - 4b$ is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $100m + n$.

Proposed by Justin Stevens

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5 Compute the 100^{th} smallest positive integer n that satisfies the three congruences

$$\left\lfloor \frac{n}{8} \right\rfloor \equiv 3 \pmod{4},$$
$$\left\lfloor \frac{n}{32} \right\rfloor \equiv 2 \pmod{4},$$
$$\left\lfloor \frac{n}{256} \right\rfloor \equiv 1 \pmod{4}.$$

Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Proposed by Michael Tang

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6 Emma's calculator has ten buttons: one for each digit $1, 2, \ldots, 9$, and one marked "clear". When Emma presses one of the buttons marked with a digit, that digit is appended to the right of the display. When she presses the "clear" button, the display is completely erased. If Emma starts with an empty display and presses five (not necessarily distinct) buttons at random, where all ten buttons have equal probability of being chosen, the expected value of the number produced is $\frac{m}{n}$, for relatively prime positive integers m and n. Find $100m + n$. (Take an empty display to represent the number 0.)

Proposed by Michael Tang

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7 Let $p = 2017$ be a prime. Find the remainder when

$$\left\lfloor \frac{1^p}{p} \right\rfloor + \left\lfloor \frac{2^p}{p} \right\rfloor + \left\lfloor \frac{3^p}{p} \right\rfloor + \cdots + \left\lfloor \frac{2015^p}{p} \right\rfloor$$

is divided by p. Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Proposed by David Altizio

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8 Let $\triangle ABC$ be an equilateral triangle with side length s and P a point in the interior of this triangle. Suppose that PA, PB, and PC are the roots of the polynomial $t^3 - 18t^2 + 91t - 89$. Then s^2 can be written in the form $m + \sqrt{n}$ where m and n are positive integers. Find $100m + n$.

Proposed by David Altizio

Day 24 February 22, 2016

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1 Suppose $a_1, a_2, a_3, \ldots$ is an arithmetic sequence such that

$$a_1 + a_2 + a_3 + \cdots + a_{48} + a_{49} = 1421.$$

Find the value of $a_1 + a_4 + a_7 + a_{10} + \cdots + a_{49}$.

Proposed by Tony Kim

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2 Michael, David, Evan, Isabella, and Justin compete in the NIMO Super Bowl, a round-robin cereal-eating tournament. Each pair of competitors plays exactly one game, in which each competitor has an equal chance of winning (and there are no ties). The probability that none of the five players wins all of his/her games is $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

Proposed by Evan Chen

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3 Let f be the quadratic function with leading coefficient 1 whose graph is tangent to that of the lines $y = -5x + 6$ and $y = x - 1$. The sum of the coefficients of f is $\frac{p}{q}$, where p and q are positive relatively prime integers. Find $100p + q$.

Proposed by David Altizio

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4 Justine has two fair dice, one with sides labeled $1, 2, \ldots, m$ and one with sides labeled $1, 2, \ldots, n$. She rolls both dice once. If $\frac{3}{20}$ is the probability that at least one of the numbers showing is at most 3, find the sum of all distinct possible values of $m + n$.

Proposed by Justin Stevens

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5 A wall made of mirrors has the shape of $\triangle ABC$, where $AB = 13$, $BC = 16$, and $CA = 9$. A laser positioned at point A is fired at the midpoint M of BC. The shot reflects about BC and then strikes point P on AB. If $\frac{AM}{MP} = \frac{m}{n}$ for relatively prime positive integers m, n, compute $100m + n$.

Proposed by Michael Tang

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6 Let S be the sum of all positive integers that can be expressed in the form $2^a \cdot 3^b \cdot 5^c$, where a, b, c are positive integers that satisfy $a + b + c = 10$. Find the remainder when S is divided by 1001.

Proposed by Michael Ren

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7 Let A and B be points with $AB = 12$. A point P in the plane of A and B is *special* if there exist points X, Y such that

-P lies on segment XY,
-$PX : PY = 4 : 7$, and
-the circumcircles of AXY and BXY are both tangent to line AB.

A point P that is not special is called *boring*.
Compute the smallest integer n such that any two boring points have distance less than $\sqrt{n/10}$ from each other.

Proposed by Michael Ren

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8 For a complex number $z \neq 3$,4, let $F(z)$ denote the real part of $\frac{1}{(3-z)(4-z)}$. If

$$\int_0^1 F\left(\frac{\cos 2\pi t + i \sin 2\pi t}{5}\right)\, dt = \frac{m}{n}$$

for relatively prime positive integers m and n, find $100m + n$.

Proposed by Evan Chen

Day 25 April 5, 2016

1 In quadrilateral $ABCD$, $AB \parallel CD$ and $BC \perp AB$. Lines AC and BD intersect at E. If $AB = 20$, $BC = 2016$, and $CD = 16$, find the area of $\triangle BCE$.

Proposed by Harrison Wang

2 A time is chosen randomly and uniformly in an 24-hour day. The probability that at that time, the (non-reflex) angle between the hour hand and minute hand on a clock is less than $\frac{360}{11}$ degrees is $\frac{m}{n}$ for coprime positive integers m and n. Find $100m + n$.

Proposed by Yannick Yao

3 David, Kevin, and Michael each choose an integer from the set $\{1, 2, \ldots, 100\}$ randomly, uniformly, and independently of each other. The probability that the positive difference between David's and Kevin's numbers is *strictly* less than that of Kevin's and Michael's numbers is $\frac{m}{n}$, for coprime positive integers m and n. Find $100m + n$.

Proposed by Richard Yi

4 Let S be the set of all pairs of positive integers (x, y) for which $2x^2 + 5y^2 \le 5 + 6xy$. Compute $\displaystyle\sum_{(x,y)\in S} (x + y + 100)$.

Proposed by Daniel Whatley

5 Bob starts with an empty whiteboard. He then repeatedly chooses one of the digits $1, 2, \ldots, 9$ (uniformly at random) and appends it to the end of the currently written number. Bob stops when the number on the board is a multiple of 25. Let E be the expected number of digits that Bob writes. If $E = \frac{m}{n}$ for relatively prime positive integers m and n, find $100m + n$.

Proposed by Amogh Gaitonde

6 As a reward for working for NIMO, Evan divides 100 indivisible marbles among three of his volunteers: David, Justin, and Michael. (Of course, each volunteer must get at least one marble!) However, Evan knows that, in the middle of the night, Lewis will select a positive integer $n > 1$ and, for each volunteer, steal exactly $\frac{1}{n}$ of his marbles (if possible, i.e. if n divides the number of marbles). In how many ways can Evan distribute the 100 marbles so that Lewis is unable to steal marbles from every volunteer, regardless of which n he selects?

Proposed by Jack Cornish

7 Let $(a_1, a_2, \ldots, a_{13})$ be a permutation of $(1, 2, \ldots, 13)$. Ayvak takes this permutation and makes a series of *moves*, each of which consists of choosing an integer i from 1 to 12, inclusive, and swapping the positions of a_i and a_{i+1}. Define the *weight* of a permutation to be the minimum number of moves Ayvak needs to turn it into $(1, 2, \dots, 13)$.

The arithmetic mean of the weights of all permutations $(a_1, \dots, a_{13})$ of $(1, 2, \dots, 13)$ for which $a_5 = 9$ is $\frac{m}{n}$, for coprime positive integers m and n. Find $100m + n$.

Proposed by Alex Gu

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8 Rectangle $EFGH$ with side lengths 8, 9 lies inside rectangle $ABCD$ with side lengths 13, 14, with their corresponding sides parallel. Let $\ell_A, \ell_B, \ell_C, \ell_D$ be the lines connecting A, B, C, D, respectively, with the vertex of $EFGH$ closest to them. Let $P = \ell_A \cap \ell_B$, $Q = \ell_B \cap \ell_C$, $R = \ell_C \cap \ell_D$, and $S = \ell_D \cap \ell_A$. Suppose that the greatest possible area of quadrilateral $PQRS$ is $\frac{m}{n}$, for relatively prime positive integers m and n. Find $100m + n$.

Proposed by Yannick Yao

Day 26 May 12, 2016

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1 Three congruent circles of radius 2 are drawn in the plane so that each circle passes through the centers of the other two circles. The region common to all three circles has a boundary consisting of three congruent circular arcs. Let K be the area of the triangle whose vertices are the midpoints of those arcs. If $K = \sqrt{a} - b$ for positive integers a, b, find $100a + b$.

Proposed by Michael Tang

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2 Find the greatest positive integer n such that 2^n divides

$$\text{lcm}\left(1^1, 2^2, 3^3, \dots, 2016^{2016}\right).$$

Proposed by Michael Tang

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3 A round-robin tournament has six competititors. Each round between two players is equally likely to result in a win for a given player, a loss for that player, or a tie. The results of the tournament are *nice* if for all triples of distinct players (A, B, C),

1. If A beat B and B beat C, then A also beat C;

2. If A and B tied, then either C beat both A and B, or C lost to both A and B.

The probability that the results of the tournament are *nice* is $p = \frac{m}{n}$, for coprime positive integers m and n. Find m.

Proposed by Michael Tang

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4 Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Let ω_A, ω_B and ω_C be circles such that ω_B and ω_C are tangent at A, ω_C and ω_A are tangent at B, and ω_A and ω_B are tangent at C. Suppose that line AB intersects ω_B at a point $X \neq A$ and line AC intersects ω_C at a point $Y \neq A$. If lines XY and BC intersect at P, then $\frac{BC}{BP} = \frac{m}{n}$ for coprime positive integers m and n. Find $100m + n$.

Proposed by Michael Ren

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5 The equation $x^3 - 3x^2 - 7x - 1 = 0$ has three distinct real roots a, b, and c. If

$$\left(\frac{1}{\sqrt[3]{a} - \sqrt[3]{b}} + \frac{1}{\sqrt[3]{b} - \sqrt[3]{c}} + \frac{1}{\sqrt[3]{c} - \sqrt[3]{a}}\right)^2 = \frac{p\sqrt[3]{q}}{r}$$

where p, q, r are positive integers such that $\gcd(p, r) = 1$ and q is not divisible by the cube of a prime, find $100p + 10q + r$.

Proposed by Michael Tang and David Altizio
