1995 Israel Mathematical Olympiad

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1 Solve the system $x+\log\left(x+\sqrt{x^2+1}\right)=y$ $y+\log\left(y+\sqrt{y^2+1}\right)=z$ $z+\log\left(z+\sqrt{z^2+1}\right)=x$

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2 Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that $AB\perp PQ$ and is tangent to O. Prove that AC bisects $\angle PAB$

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3 If k and n are positive integers, prove the inequality

$$\frac{1}{kn}+\frac{1}{kn+1}+...+\frac{1}{(k+1)n-1}\ge n\left(\sqrt[n]{\frac{k+1}{k}}-1\right)$$

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4 Find all integers m and n satisfying $m^3-n^3-9mn=27$.

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5 Let n be an odd positive integer and let $x_1,x_2,...,x_n$ be n distinct real numbers that satisfy $|x_i-x_j|\le 1$ for $1\le i<j\le n$. Prove that

$$\sum_{i<j}|x_i-x_j|\le\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]-1\right)$$

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6 A 1995×1995 square board is given. A coloring of the cells of the board is called *good* if the cells can be rearranged so as to produce a colored square board that is symmetric with respect to the main diagonal. Find all values of k for which any k-coloring of the given board is *good*.

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7 For certain n countries there is an airline connecting any two countries, but some of the airlines are closed. Show that if the number of the closed airlines does not exceed $n-3$, then one can make a round trip using the remaining airlines, starting from one of the countries, visiting every country exactly once and returning to the starting country.

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8 A real number α is given. Find all functions $f:R^+\to R^+$ satisfying $\alpha x^2f\left(\frac{1}{x}\right)+f(x)=\frac{x}{x+1}$ for all $x>0$.
