All-Russian Olympiad 2015

www.artofproblemsolving.com/community/c255746

by gavrilos, IstekOlympiadTeam, sqing, silouan, utkarshgupta, Mukhammadiev

–	Grade 9
–	Day 1

1   Real numbers a and b are chosen so that each of two quadratic trinomials $x^2 + ax + b$ and $x^2 + bx + a$ has two distinct real roots,and the product of these trinomials has exactly three distinct real roots.Determine all possible values of the sum of these three roots. *(S.Berlov)*

3   Let a, x, y be positive integer such that $a > 100, x > 100, y > 100$ and $y^2 - 1 = a^2(x^2 - 1)$. Find the minimum value of $\frac{a}{x}$.

–	Day 2

5 100 integers are arranged in a circle. Each number is greater than the sum of the two subsequent numbers (in a clockwise order). Determine the maximal possible number of positive numbers in such circle. *(S.Berlov)*

6   A field has a shape of checkboard 41x41 square. A tank concealed in one of the cells of the field. By one shot, a fighter airplane fires one of the cells. If a shot hits the tank, then the tank moves to a neighboring cell of the field, otherwise it stays in its cell (the cells are neighbours if they share a side). A pilot has no information about the tank , one needs to hit it twice. Find the least number of shots sufficient to destroy the tank for sure. *(S.Berlov,A.Magazinov)*

7   An acute-angled ABC $(AB < AC)$ is inscribed into a circle ω. Let M be the centroid of ABC, and let AH be an altitude of this triangle. A ray MH meets ω at A'. Prove that the circumcircle of the triangle $A'HB$ is tangent to AB. *(A.I. Golovanov , A.Yakubov)*

8 $N \geq 9$ distinct real numbers are written on a blackboard. All these numbers are nonnegative, and all are less than 1. It happens that for very 8 distinct numbers on the board, the board contains the ninth number distinct from eight such that the sum of all these nine numbers is integer. Find all values N for which this is possible. *(F. Nilov)*

–	Grade 10
–	Day 1

1   We say that a positive integer is an *almost square*, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.
V. Senderov

2   Given is a parallelogram $ABCD$, with $AB < AC < BC$. Points E and F are selected on the circumcircle ω of ABC so that the tangenst to ω at these points pass through point D and the segments AD and CE intersect.
It turned out that $\angle ABF = \angle DCE$. Find the angle $\angle ABC$.
A. Yakubov, S. Berlov

4   We denote by $S(k)$ the sum of digits of a positive integer number k. We say that the positive integer a is n-good, if there is a sequence of positive integers $a_0, a_1, \ldots, a_n$, so that $a_n = a$ and $a_{i+1} = a_i - S(a_i)$ for all $i = 0, 1, ..., n-1$.
Is it true that for any positive integer n there exists a positive integer b, which is n-good, but not $(n+1)$-good?
A. Antropov

–   Day 2

5   It is known that a cells square can be cut into n equal figures of k cells.
Prove that it is possible to cut it into k equal figures of n cells.

7   In an acute-angled and not isosceles triangle ABC, we draw the median AM and the height AH.
Points Q and P are marked on the lines AB and AC, respectively, so that the $QM \perp AC$ and $PM \perp AB$.
The circumcircle of PMQ intersects the line BC for second time at point X. Prove that $BH = CX$.
M. Didin

–   Grade 11

–   Day 1

1   Parallelogram $ABCD$ is such that angle $B < 90$ and $AB < BC$. Points E and F are on the circumference of ω inscribing triangle ABC, such that tangents to ω in those points pass through D. If $\angle EDA = \angle FDC$, find $\angle ABC$.

2   Let $n > 1$ be a natural number. We write out the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$ such that they are all in their simplest form. Let the sum of the numerators be $f(n)$. For what $n > 1$ is one of $f(n)$ and $f(2015n)$ odd, but the other is even?

3   110 teams participate in a volleyball tournament. Every team has played every other team exactly once (there are no ties in volleyball). Turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

4   You are given N such that $n \geq 3$. We call a set of N points on a plane acceptable if their abscissae are unique, and each of the points is coloured either red or blue. Let's say that a polynomial $P(x)$ divides a set of acceptable points either if there are no red dots above the graph of $P(x)$, and below, there are no blue dots, or if there are no blue dots above the graph of $P(x)$ and there are no red dots below. Keep in mind, dots of both colors can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary set of N points divisible by a polynomial of degree k?

–   Day 2

5   An immortal flea jumps on whole points of the number line, beginning with 0. The length of the first jump is 3, the second 5, the third 9, and so on. The length of kth jump is equal to $2^k + 1$. The flea decides whether to jump left or right on its own. Is it possible that sooner or later the flee will have been on every natural point, perhaps having visited some of the points more than once?

6   Let a,b,c,d be real numbers satisfying $|a|, |b|, |c|, |d| > 1$ and $abc+abd+acd+bcd+a+b+c+d = 0$. Prove that $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} > 0$

7   A scalene triangle ABC is inscribed within circle ω. The tangent to the circle at point C intersects line AB at point D. Let I be the center of the circle inscribed within $\triangle ABC$. Lines AI and BI intersect the bisector of $\angle CDB$ in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circle ω.

8   Given natural numbers a and b, such that $a < b < 2a$. Some cells on a graph are colored such that in every rectangle with dimensions $A \times B$ or $B \times A$, at least one cell is colored. For which greatest α can you say that for every natural number N you can find a square $N \times N$ in which at least $\alpha \cdot N^2$ cells are colored?