Finals 2016

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–	Day 1
1	Let p be a certain prime number. Find all non-negative integers n for which polynomial $P(x)=x^4-2(n+p)x^2+(n-p)^2$ may be rewritten as product of two quadratic polynomials $P_1,\ P_2\in\mathbb{Z}[X]$.
2	Let $ABCD$ be a quadrilateral circumscribed on the circle ω with center I. Assume $\angle BAD+\angle ADC<\pi$. Let $M,\ N$ be points of tangency of ω with $AB,\ CD$ respectively. Consider a point $K\in MN$ such that $AK=AM$. Prove that ID bisects the segment KN.
3	Let $a,\ b\in\mathbb{Z}_+$. Denote $f(a,b)$ the number sequences $s_1,\ s_2,\ ...,\ s_a$, $s_i\in\mathbb{Z}$ such that $\lvert s_1\rvert+\lvert s_2\rvert+...+\lvert s_a\rvert\le b$. Show that $f(a,b)=f(b,a)$.
–	Day2
4	Let k,n be odd positve integers greater than 1. Prove that if there a exists natural number a such that $k\mid 2^a+1,\ n\mid 2^a-1$, then there is no natural number b satisfying $k\mid 2^b-1,\ n\mid 2^b+1$.
5	There are given two positive real number $a<b$. Show that there exist positive integers $p,\ q,\ r,\ s$ satisfying following conditions: $1.\ a<\frac{p}{q}<\frac{r}{s}<b.\ 2.\ p^2+q^2=r^2+s^2$.
6	Let I be an incenter of $\triangle ABC$. Denote $D,\ S\neq A$ intersections of AI with $BC,\ O(ABC)$ respectively. Let $K,\ L$ be incenters of $\triangle DSB,\ \triangle DCS$. Let P be a reflection of I with the respect to KL. Prove that $BP\perp CP$.