Turkey Team Selection Test 2016

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Day 1 April 2nd

1 In an acute triangle ABC, a point P is taken on the A-altitude. Lines BP and CP intersect the sides AC and AB at points D and E, respectively. Tangents drawn from points D and E to the circumcircle of triangle BPC are tangent to it at points K and L, respectively, which are in the interior of triangle ABC. Line KD intersects the circumcircle of triangle AKC at point M for the second time, and line LE intersects the circumcircle of triangle ALB at point N for the second time. Prove that

$$\frac{KD}{MD}=\frac{LE}{NE} \iff \text{Point P is the orthocenter of triangle ABC}$$

2 In a class with 23 students, each pair of students have watched a movie together. Let the set of movies watched by a student be his *movie collection*. If every student has watched every movie at most once, at least how many different movie collections can these students have?

3 Let a,b,c be non-negative real numbers such that $a^2+b^2+c^2\le 3$ then prove that;

$$(a+b+c)(a+b+c-abc)\ge 2(a^2b+b^2c+c^2a)$$

Day 2 April 3rd

4 A sequence of real numbers $a_0, a_1, \ldots$ satisfies the condition

$$\sum_{n=0}^{m} a_n\cdot(-1)^n\cdot\binom{m}{n}=0$$

for all large enough positive integers m. Prove that there exists a polynomial P such that $a_n=P(n)$ for all $n\ge 0$.

5 Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for all $m,n\in\mathbb{N}$ holds $f(mn)=f(m)f(n)$ and $m+n\mid f(m)+f(n)$.

6 In a triangle ABC with $AB=AC$, let D be the midpoint of $[BC]$. A line passing through D intersects AB at K, AC at L. A point E on $[BC]$ different from D, and a point P on AE is taken such that $\angle KPL = 90^\circ - \frac{1}{2}\angle KAL$ and E lies between A and P. The circumcircle of triangle PDE intersects PK at point X, PL at point Y for the second time. Lines DX and AB intersect at M, and lines DY and AC intersect at N. Prove that the points P, M, A, N are concyclic.

Day 3 April 4th

7 $A_1, A_2, \ldots A_k$ are different subsets of the set $\{1, 2, \ldots, 2016\}$. If $A_i \cap A_j$ forms an arithmetic sequence for all $1 \le i < j \le k$, what is the maximum value of k?

8 All angles of the convex n-gon $A_1A_2\ldots A_n$ are obtuse, where $n \ge 5$. For all $1 \le i \le n$, O_i is the circumcenter of triangle $A_{i-1}A_iA_{i+1}$ (where $A_0 = A_n$ and $A_{n+1} = A_1$). Prove that the closed path $O_1O_2\ldots O_n$ doesn't form a convex n-gon.

9 p is a prime. Let K_p be the set of all polynomials with coefficients from the set $\{0, 1, \ldots, p-1\}$ and degree less than p. Assume that for all pairs of polynomials $P, Q \in K_p$ such that $P(Q(n)) \equiv n \pmod{p}$ for all integers n, the degrees of P and Q are equal. Determine all primes p with this condition.