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by mathematics2004, mathisreal

1 For every $0 < \alpha < 1$, let $R(\alpha)$ be the region in $\mathbb{R}^2$ whose boundary is the convex pentagon of vertices $(0, 1-\alpha), (\alpha, 0), (1, 0), (1, 1)$ and $(0, 1)$. Let R be the set of points that belong simultaneously to each of the regions $R(\alpha)$ with $0 < \alpha < 1$, that is, $R = \bigcap_{0<\alpha<1} R(\alpha)$.

Determine the area of R.

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2 Let $r > s$ be positive integers. Let $P(x)$ and $Q(x)$ be distinct polynomials with real coefficients, non-constant(s), such that $P(x)^r - P(x)^s = Q(x)^r - Q(x)^s$ for every $x \in \mathbb{R}$.
Prove that $(r, s) = (2, 1)$.

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3 Let m, n and N be positive integers and $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$ a set of residues modulo N. Consider a table $m \times n$ such that each one of the mn cells has an element of $\mathbb{Z}_N$. A *move* is choose an element $g \in \mathbb{Z}_N$, a cell in the table and add $+g$ to the elements in the same row/column of the chosen cell(the sum is modulo N). Prove that if N is coprime with $m-1, n-1, m+n-1$ then any initial arrangement of your elements in the table cells can become any other arrangement using an finite quantity of moves.

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4 Let $\mathbb{Z}^+$ be the set of positive integers.
a) Prove that there is only one function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, strictly increasing, such that $f(f(n)) = 2n+1$ for every $n \in \mathbb{Z}^+$.
b) For the function in **a**. Prove that for every $n \in \mathbb{Z}^+$ $\frac{4n+1}{3} \leq f(n) \leq \frac{3n+1}{2}$
c) Prove that in each inequality side of **b** the equality can reach by infinite positive integers n.

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5 For every positive integer n, let $s(n)$ be the sum of the exponents of 71 and 97 in the prime factorization of n; for example, $s(2021) = s(43 \cdot 47) = 0$ and $s(488977) = s(71^2 \cdot 97) = 3$. If we define $f(n) = (-1)^{s(n)}$, prove that the limit

$$\lim_{n \to +\infty} \frac{f(1) + f(2) + \cdots + f(n)}{n}$$

exists and determine its value.

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6 Let $0 \leq a < b$ be real numbers. Prove that there is no continuous function $f : [a, b] \to \mathbb{R}$ such that

$$\int_a^b f(x)x^{2n} \mathrm{d}x > 0 \quad \text{and} \quad \int_a^b f(x)x^{2n+1} \mathrm{d}x < 0$$

for every integer $n \geq 0$.
