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by Rickyminer, mathisreal

1 Let $\alpha > 1$ and consider the function $f(x) = x^\alpha$ for $x \geq 0$. For $t > 0$, define $M(t)$ as the largest area that a triangle with vertices $(0,0), (s, f(s)), (t, f(t))$ could reach, for $s \in (0,t)$. Let $A(t)$ be the area of the region bounded by the segment with endpoints $(0,0)$,$(t, f(t))$ and the graph of $y = f(x)$.
(a) Show that $A(t)/M(t)$ does not depend on t. We denote this value by $c(\alpha)$. Find $c(\alpha)$.
(b) Determine the range of values of $c(\alpha)$ when α varies in the interval $(1, +\infty)$.

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2 Find all triples of positive integers (a, b, c) such that the following equations are both true:
I- $a^2 + b^2 = c^2$
II- $a^3 + b^3 + 1 = (c-1)^3$

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3 Let (m, r, s, t) be positive integers such that $m \geq s+1$ and $r \geq t$. Consider m sets $A_1, A_2, \dots, A_m$ with r elements each one. Suppose that, for each $1 \leq i \leq m$, there exist at least t elements of A_i, such that each one(element) belongs to (at least) s sets A_j where $j \neq i$. Determine the greatest quantity of elements in the following set $A_1 \cup A_2 \cup A_3 \cdots \cup A_m$.

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4 For each polynomial $P(x)$ with real coefficients, define $P_0 = P(0)$ and $P_j(x) = x^j \cdot P^{(j)}(x)$ where $P^{(j)}$ denotes the j-th derivative of P for $j \geq 1$.
Prove that there exists one unique sequence of real numbers $b_0, b_1, b_2, \dots$ such that for each polynomial $P(x)$ with real coefficients and for each x real, we have $P(x) = b_0 P_0 + \sum_{k \geq 1} b_k P_k(x) = b_0 P_0 + b_1 P_1(x) + b_2 P_2(x) + \dots$

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5 Determine all the positive real numbers $x_1, x_2, x_3, \dots, x_{2021}$ such that $x_{i+1} = \frac{x_i^3 + 2}{3x_i^2}$ for every $i = 1, 2, 3, \dots, 2020$ and $x_{2021} = x_1$

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6 For a set A, we define $A + A = \{a + b : a, b \in A\}$. Determine whether there exists a set A of positive integers such that

$$\sum_{a \in A} \frac{1}{a} = +\infty \quad \text{and} \quad \lim_{n \to +\infty} \frac{|(A+A) \cap \{1, 2, \cdots, n\}|}{n} = 0.$$

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