USA TST Selection Test 2021

www.artofproblemsolving.com/community/c2582544

by nukelauncher, pad, AwesomeYRY

Day 1 November 4, 2021

1 Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

Holden Mui

2 Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots.$$

Merlijn Staps

3 Find all positive integers $k > 1$ for which there exists a positive integer n such that $\binom{n}{k}$ is divisible by n, and $\binom{n}{m}$ is not divisible by n for $2 \leq m < k$.

Merlijn Staps

Day 2 December 9, 2021

4 Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

Holden Mui

5 Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges. A tree is a connected graph with no cycles. A leaf is a vertex of degree 1

Vincent Huang

6 Triangles ABC and DEF share circumcircle Ω and incircle ω so that points $A, F, B, D, C,$ and E occur in this order along Ω. Let Δ_A be the triangle formed by lines AB, AC, and EF, and define triangles $\Delta_B, \Delta_C, \ldots, \Delta_F$ similarly. Furthermore, let Ω_A and ω_A be the circumcircle and incircle of triangle Δ_A, respectively, and define circles $\Omega_B, \omega_B, \ldots, \Omega_F, \omega_F$ similarly.

(a) Prove that the two common external tangents to circles Ω_A and Ω_D and the two common external tangents to ω_A and ω_D are either concurrent or pairwise parallel.

(b) Suppose that these four lines meet at point T_A, and define points T_B and T_C similarly. Prove that points T_A, T_B, and T_C are collinear.

Nikolai Beluhov

Day 3 January 13, 2022

---

7 Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0,0)$ and ending at a point on the line $x+y=n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths. A lattice point is a point (x,y) where x and y are integers. A path of lattice points with length n is a sequence of lattice points $P_0, P_1, \dots, P_n$ in which any two adjacent points in the sequence have distance 1 from each other.

Ankit Bisain and Holden Mui

---

8 Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

Fedir Yudin

---

9 Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1\le k\le q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n}$$

is not an integer. (The sum is over all $1 \le k \le q$ with p not dividing k.)

Victor Wang
