

USAJMO 2016

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Day 1 April 19th

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- 1** The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω . Let P be a variable point on the arc \widehat{BC} that does not contain A , and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

Prove that as P varies, the circumcircle of triangle $\triangle PI_B I_C$ passes through a fixed point.

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- 2** Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

Proposed by Evan Chen

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- 3** Let X_1, X_2, \dots, X_{100} be a sequence of mutually distinct nonempty subsets of a set S . Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S , that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S .

Day 2 April 20th

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- 4** Find, with proof, the least integer N such that if any 2016 elements are removed from the set $1, 2, \dots, N$, one can still find 2016 distinct numbers among the remaining elements with sum N .

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- 5** Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line \overleftrightarrow{BC} , and points P and Q are the feet of the perpendiculars from H to the lines \overleftrightarrow{AB} and \overleftrightarrow{AC} , respectively.

Given that

$$AH^2 = 2 \cdot AO^2,$$

prove that the points O, P , and Q are collinear.

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- 6** Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all real numbers x and y ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

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