National Science Olympiad 2021

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–	Day 1
1	On the whiteboard, the numbers are written sequentially: $1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$. Andi has to paste a $+$ (plus) sign or $-$ (minus) sign in between every two successive numbers, and compute the value. Determine the least odd positive integer that Andi can't get from this process.
2	Let ABC be an acute triangle. Let D and E be the midpoint of segment AB and AC respectively. Suppose L_1 and L_2 are circumcircle of triangle ABC and ADE respectively. CD intersects L_1 and L_2 at $M(M \neq C)$ and $N(N \neq D)$. If $DM = DN$, prove that $\triangle ABC$ is isosceles.
3	A natural number is called a *prime power* if that number can be expressed as p^n for some prime p and natural number n. Determine the largest possible n such that there exists a sequence of prime powers $a_1, a_2, \dots, a_n$ such that $a_i = a_{i-1} + a_{i-2}$ for all $3 \leq i \leq n$.
4	Let x, y and z be positive reals such that $x + y + z = 3$. Prove that $$2\sqrt{x+\sqrt{y}}+2\sqrt{y+\sqrt{z}}+2\sqrt{z+\sqrt{x}} \leq \sqrt{8+x-y}+\sqrt{8+y-z}+\sqrt{8+z-x}$$
–	Day 2
5	Let $P(x) = x^2 + rx + s$ be a polynomial with real coefficients. Suppose $P(x)$ has two distinct real roots, both of which are less than -1 and the difference between the two is less than 2. Prove that $P(P(x)) > 0$ for all real x.
6	There are n natural numbers written on the board. Every move, we could erase a, b and change it to $\gcd(a,b)$ and lcm$(a,b) - \gcd(a,b)$. Prove that in finite number of moves, all numbers in the board could be made to be equal.
7	Given $\triangle ABC$ with circumcircle ℓ. Point M in $\triangle ABC$ such that AM is the angle bisector of $\angle BAC$. Circle with center M and radius MB intersects ℓ and BC at D and E respectively, $(B \neq D, B \neq E)$. Let P be the midpoint of arc BC in ℓ that didn't have A. Prove that AP angle bisector of $\angle DPE$ if and only if $\angle B = 90^\circ$.
8	On a 100×100 chessboard, the plan is to place several 1×3 boards and 3×1 board, so that

- Each tile of the initial chessboard is covered by at most one small board.
- The boards cover the entire chessboard tile, except for one tile.
- The sides of the board are placed parallel to the chessboard.

Suppose that to carry out the instructions above, it takes H number of 1×3 boards and V number of 3×1 boards. Determine all possible pairs of (H, V).