Mexico National Olympiad 2021

www.artofproblemsolving.com/community/c2609131

by parmenides51, jampm, Jjesus

– Day 1

1 The real positive numbers a_1, a_2, a_3 are three consecutive terms of an arithmetic progression, and similarly, b_1, b_2, b_3 are distinct real positive numbers and consecutive terms of an arithmetic progression. Is it possible to use three segments of lengths a_1, a_2, a_3 as bases, and other three segments of lengths b_1, b_2, b_3 as altitudes, to construct three rectangles of equal area ?

2 Let ABC be a triangle with $\angle ACB > 90^\circ$, and let D be a point on BC such that AD is perpendicular to BC. Consider the circumference Γ with with diameter BC. A line ℓ passes through D and is tangent to Γ at P, cuts AC at M (such that M is in between A and C), and cuts the side AB at N. Prove that M is the midpoint of DP if and only if N is the midpoint of AB.

– Day 2

4 Let ABC be an acutangle scalene triangle with $\angle BAC = 60^\circ$ and orthocenter H. Let ω_b be the circumference passing through H and tangent to AB at B, and ω_c the circumference passing through H and tangent to AC at C.

- Prove that ω_b and ω_c only have H as common point.
- Prove that the line passing through H and the circumcenter O of triangle ABC is a common tangent to ω_b and ω_c.

Note: The orthocenter of a triangle is the intersection point of the three altitudes, whereas the circumcenter of a triangle is the center of the circumference passing through it's three vertices.

5 If $n = \overline{a_1a_2\cdots a_{k-1}a_k}$, be $s(n)$ such that
. If k is even, $s(n) = \overline{a_1a_2} + \overline{a_3a_4}\cdots + \overline{a_{k-1}a_k}$
. If k is odd, $s(n) = a_1 + \overline{a_2a_3}\cdots + \overline{a_{k-1}a_k}$
For example $s(123) = 1 + 23 = 24$ and $s(2021) = 20 + 21 = 41$
Be n is $digital$ if $s(n)$ is a divisor of n. Prove that among any 198 consecutive positive integers, all of them less than 2000021 there is one of them that is $digital$.

6 Determine all non empty sets $C_1, C_2, C_3, \cdots$ such that each one of them has a finite number of elements, all their elements are positive integers, and they satisfy the following property: For any positive integers n and m, the number of elements in the set C_n plus the number of elements in the set C_m equals the sum of the elements in the set C_{m+n}.

Note: We denote $|C_n|$ the number of elements in the set C_n, and S_k as the sum of the elements in the set C_n so the problem's condition is that for every n and m:

$$|C_n| + |C_m| = S_{n+m}$$

is satisfied.