Olympic Revenge 2021

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by jpedrorvc1

1 Let a, b, c, k be positive reals such that $ab+bc+ca \leq 1$ and $0 < k \leq \frac{9}{2}$. Prove that:

$$\sqrt[3]{\frac{k}{a}+(9-3k)b}+\sqrt[3]{\frac{k}{b}+(9-3k)c}+\sqrt[3]{\frac{k}{c}+(9-3k)a} \leq \frac{1}{abc}.$$

Proposed by Zhang Yanzong and Song Qing

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2 Evan is a n-dimensional being that lives in a house formed by the points of $\mathbb{Z}^n_{\geq 0}$. His room is the set of points in which coordinates are all less than or equal to 2021. Evan's room has been infested with bees, so he decides to flush them out through *captures*. A *capture* can be performed by eliminating a bee from point $(a_1, a_2, \dots, a_n)$ and replacing it with n bees, one in each of the points:

$$(a_1+1, a_2, \dots, a_n), (a_1, a_2+1, \dots, a_n), \dots, (a_1, a_2, \dots, a_n+1)$$

However, two bees can never occupy the same point in the house. Determine, for every n, the greatest value $A(n)$ of bees such that, for some initial arrangement of these bees in Evan's room, he is able to accomplish his goal with a finite amount of *captures*.

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3 Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

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4 On a chessboard, Po controls a white queen and plays, in alternate turns, against an invisible black king (there are only those two pieces on the board). The king cannot move to a square where he would be in check, neither capture the queen. Every time the king makes a move, Po receives a message from beyond that tells which direction the king has moved (up, right, up-right, etc). His goal is to make the king unable to make a movement.

Can Po reach his goal with at most 150 moves, regardless the starting position of the pieces?

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5 Prove there aren't positive integers a, b, c, d forming an arithmetic progression such that $ab+1, ac+1, ad+1, bc+1, bd+1, cd+1$ are all perfect squares.
