KJMO 2021

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–	day 1
1	For positive integers n, k, r, denote by $A(n, k, r)$ the number of integer tuples $(x_1, x_2, \dots, x_k)$ satisfying the following conditions. - $x_1 \ge x_2 \ge \cdots \ge x_k \ge 0$ - $x_1 + x_2 + \cdots + x_k = n$ - $x_1 - x_k \le r$ For all positive integers $s, t \ge 2$, prove that $$A(st, s, t) = A(s(t-1), s, t) = A((s-1)t, s, t).$$
2	Let $\{a_n\}$ be a sequence of integers satisfying the following conditions. - $a_1 = 2021^{2021}$ - $0 \le a_k < k$ for all integers $k \ge 2$ - $a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{k+1}a_k$ is multiple of k for all positive integers k. Determine the 2021^{2022}th term of the sequence $\{a_n\}$.
3	Let $ABCD$ be a cyclic quadrilateral with circumcircle Ω and let diagonals AC and BD intersect at X. Suppose that $AEFB$ is inscribed in a circumcircle of triangle ABX such that EF and AB are parallel. FX meets the circumcircle of triangle CDX again at G. Let EX meets AB at P, and XG meets CD at Q. Denote by S the intersection of the perpendicular bisector of $\overline{EG}$ and Ω such that S is closer to A than B. Prove that line through S parallel to PQ is tangent to Ω.
–	day 2
4	In an acute triangle ABC with $\overline{AB} < \overline{AC}$, angle bisector of A and perpendicular bisector of $\overline{BC}$ intersect at D. Let P be an interior point of triangle ABC. Line CP meets the circumcircle of triangle ABP again at K. Prove that B, D, K are collinear if and only if AD and BC meet on the circumcircle of triangle APC.
5	Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $$f(f(x+y) - f(x-y)) = y^2 f(x)$$ for all $x, y \in \mathbb{R}$.

6 In a meeting of 4042 people, there are 2021 couples, each consisting of two people. Suppose that A and B, in the meeting, are friends when they know each other. For a positive integer n, each people chooses an integer from $-n$ to n so that the following conditions hold. (Two or more people may choose the same number).

- Two or less people chose 0, and if exactly two people chose 0, they are coupled.
- Two people are either coupled or don't know each other if they chose the same number.
- Two people are either coupled or know each other if they chose two numbers that sum to 0.

Determine the least possible value of n for which such number selecting is always possible.