HMMT Invitational Competition 2016

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by v_Enhance

1 Theseus starts at the point $(0,0)$ in the plane. If Theseus is standing at the point (x,y) in the plane, he can step one unit to the north to point $(x,y+1)$, one unit to the west to point $(x-1,y)$, one unit to the south to point $(x,y-1)$, or one unit to the east to point $(x+1,y)$. After a sequence of more than two such moves, starting with a step one unit to the south (to point $(0,-1)$), Theseus finds himself back at the point $(0,0)$. He never visited any point other than $(0,0)$ more than once, and never visited the point $(0,0)$ except at the start and end of this sequence of moves.

Let X be the number of times that Theseus took a step one unit to the north, and then a step one unit to the west immediately afterward. Let Y be the number of times that Theseus took a step one unit to the west, and then a step one unit to the north immediately afterward. Prove that $|X-Y|=1$.

Mitchell Lee

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2 Let ABC be an acute triangle with circumcenter O, orthocenter H, and circumcircle Ω. Let M be the midpoint of AH and N the midpoint of BH. Assume the points M, N, O, H are distinct and lie on a circle ω. Prove that the circles ω and Ω are internally tangent to each other.

Dhroova Aiylam and Evan Chen

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3 Denote by $\mathbb{N}$ the positive integers. Let $f:\mathbb{N}\to\mathbb{N}$ be a function such that, for any $w,x,y,z\in\mathbb{N}$,

$$f(f(f(z)))f(wxf(yf(z)))=z^2f(xf(y))f(w).$$

Show that $f(n!)\ge n!$ for every positive integer n.

Pakawut Jiradilok

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4 Let P be an odd-degree integer-coefficient polynomial. Suppose that $xP(x)=yP(y)$ for infinitely many pairs x,y of integers with $x\ne y$. Prove that the equation $P(x)=0$ has an integer root.

Victor Wang

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5 Let $S=\{a_1,\ldots,a_n\}$ be a finite set of positive integers of size $n\ge 1$, and let T be the set of all positive integers that can be expressed as sums of perfect powers (including 1) of distinct numbers in S, meaning

$$T=\left\{\sum_{i=1}^{n}a_i^{e_i}\mid e_1,e_2,\ldots,e_n\ge 0\right\}.$$

Show that there is a positive integer N (only depending on n) such that T contains no arithmetic progression of length N.

Yang Liu