Korea National Olympiad 2021

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– Part 1

P1 Let ABC be an acute triangle and D be an intersection of the angle bisector of A and side BC. Let Ω be a circle tangent to the circumcircle of triangle ABC and side BC at A and D, respectively. Ω meets the sides AB, AC again at E, F, respectively. The perpendicular line to AD, passing through E, F meets Ω again at G, H, respectively. Suppose that AE and GD meet at P, EH and GF meet at Q, and HD and AF meet at R. Prove that $\frac{\overline{QF}}{\overline{QG}} = \frac{\overline{HR}}{\overline{PG}}$.

P2 For positive integers n, k, r, denote by $A(n,k,r)$ the number of integer tuples $(x_1, x_2, \ldots, x_k)$ satisfying the following conditions.

- $x_1 \geq x_2 \geq \cdots \geq x_k \geq 0$
- $x_1 + x_2 + \cdots + x_k = n$
- $x_1 - x_k \leq r$

For all positive integers m, s, t, prove that

$$A(m,s,t) = A(m,t,s).$$

P3 Show that for any positive integers k and $1 \leq a \leq 9$, there exists n such that satisfies the below statement.

When $2^n = a_0 + 10a_1 + 10^2a_2 + \cdots + 10^ia_i + \cdots$ $(0 \leq a_i \leq 9$ and a_i is integer), a_k is equal to a.

– Part 2

P4 For a positive integer n, there are two countries A and B with n airports each and $n^2 - 2n + 2$ airlines operating between the two countries. Each airline operates at least one flight. Exactly one flight by one of the airlines operates between each airport in A and each airport in B, and that flight operates in both directions. Also, there are no flights between two airports in the same country. For two different airports P and Q, denote by "[i](P,Q)-travel route[/i]" the list of airports $T_0, T_1, \ldots, T_s$ satisfying the following conditions.

- $T_0 = P,\ T_s = Q$
- $T_0, T_1, \ldots, T_s$ are all distinct.
- There exists an airline that operates between the airports T_i and T_{i+1} for all $i = 0, 1, \ldots, s-1$.

Prove that there exist two airports P, Q such that there is no or exactly one [i](P, Q)-travel route[/i].

Consider a complete bipartite graph $G(A, B)$ with $|A| = |B| = n$. Suppose there are $n^2 - 2n + 2$ colors and each edge is colored by one of these colors. Define $(P, Q) - path$ a path from P to Q such that all of the edges in the path are colored the same. Prove that there exist two vertices P and Q such that there is no or only one $(P, Q) - path$.

P5 A real number sequence $a_1, \cdots, a_{2021}$ satisfies the below conditions.

$$a_1 = 1, a_2 = 2, a_{n+2} = \frac{2a_{n+1}^2}{a_n + a_{n+1}} (1 \leq n \leq 2019)$$

Let the minimum of $a_1, \cdots, a_{2021}$ be m, and the maximum of $a_1, \cdots, a_{2021}$ be M.
Let a 2021 degree polynomial

$$P(x) := (x - a_1)(x - a_2) \cdots (x - a_{2021})$$

$|P(x)|$ is maximum in $[m, M]$ when $x = \alpha$. Show that $1 < \alpha < 2$.

P6 Let ABC be an obtuse triangle with $\angle A > \angle B > \angle C$, and let M be a midpoint of the side BC. Let D be a point on the arc AB of the circumcircle of triangle ABC not containing C. Suppose that the circle tangent to BD at D and passing through A meets the circumcircle of triangle ABM again at E and $\overline{BD} = \overline{BE}$. ω, the circumcircle of triangle ADE, meets EM again at F.

Prove that lines BD and AE meet on the line tangent to ω at F.
