Croatia Team Selection Test 2016
www.artofproblemsolving.com/community/c262613
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– Day 2

Problem 1 Let $n \geq 1$ and $x_1, \dots, x_n \geq 0$. Prove that

$$(x_1 + \frac{x_2}{2} + \ldots + \frac{x_n}{n})(x_1 + 2x_2 + \ldots + nx_n) \leq \frac{(n+1)^2}{4n}(x_1 + x_2 + \ldots + x_n)^2.$$

Problem 2 Let N be a positive integer. Consider a $N \times N$ array of square unit cells. Two corner cells that lie on the same longest diagonal are colored black, and the rest of the array is white. A *move* consists of choosing a row or a column and changing the color of every cell in the chosen row or column.
What is the minimal number of additional cells that one has to color black such that, after a finite number of moves, a completely black board can be reached?

Problem 3 Let P be a point inside a triangle ABC such that

$$\frac{AP+BP}{AB} = \frac{BP+CP}{BC} = \frac{CP+AP}{CA}.$$

Lines AP, BP, CP intersect the circumcircle of triangle ABC again in A', B', C'. Prove that the triangles ABC and $A'B'C'$ have a common incircle.

Problem 4 Find all pairs (p,q) of prime numbers such that

$$p(p^2 - p - 1) = q(2q + 3).$$

– Day 3

Problem 1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y:

$$f(x^2) + xf(y) = f(x)f(x + f(y)).$$

Problem 2 Let S be a set of $N \geq 3$ points in the plane. Assume that no 3 points in S are collinear. The segments with both endpoints in S are colored in two colors.
Prove that there is a set of $N - 1$ segments of the same color which don't intersect except in their endpoints such that no subset of them forms a polygon with positive area.

Problem 3 Let ABC be an acute triangle with circumcenter O. Points E and F are chosen on segments OB and OC such that $BE = OF$. If M is the midpoint of the arc EOA and N is the midpoint of the arc AOF, prove that $\sphericalangle ENO + \sphericalangle OMF = 2\sphericalangle BAC$.

Problem 4 Let $p > 10^9$ be a prime number such that $4p + 1$ is also prime.
Prove that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \ldots, 9$.
