Pan-African Mathematical Olympiad problems from 2016

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by kron

– Day 1

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1 Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect each other at two distinct points M and N. A common tangent lines touches $\mathcal{C}_1$ at P and $\mathcal{C}_2$ at Q, the line being closer to N than to M. The line PN meets the circle $\mathcal{C}_2$ again at the point R.
Prove that the line MQ is a bisector of the angle $\angle PMR$.

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2 We have a pile of 2016 cards and a hat. We take out one card, put it in the hat and then divide the remaining cards into two arbitrary non empty piles. In the next step, we choose one of the two piles, we move one card from this pile to the hat and then divide this pile into two arbitrary non empty piles.
This procedure is repeated several times : in the k-th step $(k > 1)$ we move one card from one of the piles existing after the step $(k-1)$ to the hat and then divide this pile into two non empty piles.
Is it possible that after some number of steps we get all piles containing three cards each?

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3 For any positive integer n, we define the integer $P(n)$ by :

$P(n) = n(n+1)(2n+1)(3n+1)...(16n+1)$.

Find the greatest common divisor of the integers $P(1)$, $P(2)$, $P(3), ..., P(2016)$.

– Day 2

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4 Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$\frac{1}{(x+1)^2+y^2+1} + \frac{1}{(y+1)^2+z^2+1} + \frac{1}{(z+1)^2+x^2+1} \leq \frac{1}{2}$.

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5 Let $ABCD$ be a trapezium such that the sides AB and CD are parallel and the side AB is longer than the side CD. Let M and N be on the segments AB and BC respectively, such that each of the segments CM and AN divides the trapezium in two parts of equal area.
Prove that the segment MN intersects the segment BD at its midpoint.

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6 Consider an $n \times n$ grid formed by n^2 unit squares. We define the centre of a unit square as the intersection of its diagonals.
Find the smallest integer m such that, choosing any m unit squares in the grid, we always get four unit squares among them whose centres are vertices of a parallelogram.
