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by socrates

1 Find the greatest positive integer N with the following property: there exist integers $x_1, ..., x_N$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$.

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2 Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n.

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3 Find all functions $f : \mathbb{R} \to \mathbb{Z}$ such that

$$\left(f(f(y)-x)\right)^2 + f(x)^2 + f(y)^2 = f(y) \cdot \left(1 + 2f(f(y))\right),$$

for all $x, y \in \mathbb{R}$.

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4 A circle ω passes through the two vertices B and C of a triangle ABC. Furthermore, ω intersects segment AC in $D \neq C$ and segment AB in $E \neq B$. On the ray from B through D lies a point K such that $|BK| = |AC|$, and on the ray from C through E lies a point L such that $|CL| = |AB|$. Show that the circumcentre O of triangle AKL lies on ω.
