All-Russian Olympiad 2016

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–	Grade 9
1	A carpet dealer,who has a lot of carpets in the market,is available to exchange a carpet of dimensions $a \cdot b$ either with a carpet with dimensions $\frac{1}{a} \cdot \frac{1}{b}$ or with two carpets with dimensions $c \cdot b$ and $\frac{a}{c} \cdot b$ (the customer can select the number c).The dealer supports that,at the beginning he had a carpet with dimensions greater than 1 and,after some exchanges like the ones we described above,he ended up with a set of carpets,each one having one dimension greater than 1 and one smaller than 1.Is this possible? [i]Note:The customer can demand from the dealer to consider a carpet of dimensions $a \cdot b$ as one with dimensions $b \cdot a$.[/i]
2	ω is a circle inside angle $\measuredangle BAC$ and it is tangent to sides of this angle at B, C.An arbitrary line ℓ intersects with AB, AC at K, L,respectively and intersect with ω at P, Q.Points S, T are on BC such that $KS \parallel AC$ and $TL \parallel AB$.Prove that P, Q, S, T are concyclic.(I.Bogdanov,P.Kozhevnikov)
3	Alexander has chosen a natural number $N > 1$ and has written down in a line,and in increasing order,all his positive divisors $d_1 < d_2 < \ldots < d_s$ (where $d_1 = 1$ and $d_s = N$).For each pair of neighbouring numbers,he has found their greater common divisor.The sum of all these $s - 1$ numbers (the greatest common divisors) is equal to $N - 2$.Find all possible values of N.
5	Using each of the digits $1, 2, 3, \ldots, 8, 9$ exactly once,we form nine,not necassarily distinct,nine-digit numbers.Their sum ends in n zeroes,where n is a non-negative integer.Determine the maximum possible value of n.
6	A square is partitioned in $n^2 \geq 4$ rectanles using $2(n-1)$ lines,$n-1$ of which,are parallel to the one side of the square,$n-1$ are parallel to the other side.Prove that we can choose $2n$ rectangles of the partition,such that,for each two of them,we can place the one inside the other (possibly with rotation).
7	In triangle ABC,$AB < AC$ and ω is incirle.The A-excircle is tangent to BC at A'.Point X lies on AA' such that segment $A'X$ doesn't intersect with ω.The tangents from X to ω intersect with BC at Y, Z.Prove that the sum $XY + XZ$ not depends to point X.(Mitrofanov)
–	Grade 10

2 Diagonals AC, BD of cyclic quadrilateral $ABCD$ intersect at P.Point Q is onBC (between$B$ and C) such that $PQ \perp AC$.Prove that the line passes through the circumcenters of triangles APD and BQD is parallel to AD.(A.Kuznetsov)

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8 In acute triangle ABC,$AC < BC$,M is midpoint of AB and Ω is it's circumcircle.Let C' be antipode of C in Ω. AC' and BC' intersect with CM at K, L,respectively.The perpendicular drawn from K to AC' and perpendicular drawn from L to BC' intersect with AB and each other and form a triangle Δ.Prove that circumcircles of Δ and Ω are tangent.(M.Kungozhin)

Grade 11 Day 1

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1 There are 30 teams in NBA and every team play 82 games in the year. Bosses of NBA want to divide all teams on Western and Eastern Conferences (not necessary equally), such that number of games between teams from different conferences is half of number of all games. Can they do it?

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2 In the space given three segments A_1A_2, B_1B_2 and C_1C_2, do not lie in one plane and intersect at a point P. Let O_{ijk} be center of sphere that passes through the points A_i, B_j, C_k and P. Prove that $O_{111}O_{222}, O_{112}O_{221}, O_{121}O_{212}$ and$O_{211}O_{122}$ intersect at one point. (P.Kozhevnikov)

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3 We have sheet of paper, divided on 100×100 unit squares. In some squares we put rightangled isosceles triangles with leg =1 (Every triangle lies in one unit square and is half of this square). Every unit grid segment(boundary too) is under one leg of triangle. Find maximal number of unit squares, that don't contains triangles.

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4 There is three-dimensional space. For every integer n we build planes $x \pm y \pm z = n$. All space is divided on octahedrons and tetrahedrons.
Point (x_0, y_0, z_0) has rational coordinates but not lies on any plane. Prove, that there is such natural k , that point (kx_0, ky_0, kz_0) lies strictly inside the octahedron of partition.

Grade 11 Day 2

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5 Let n be a positive integer and let $k_0, k_1, \dots, k_{2n}$ be nonzero integers such that $k_0 + k_1 + \cdots + k_{2n} \neq 0$. Is it always possible to a permutation $(a_0, a_1, \dots, a_{2n})$ of $(k_0, k_1, \dots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0 = 0$$

has not integer roots?

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6 There are $n > 1$ cities in the country, some pairs of cities linked two-way through straight flight. For every pair of cities there is exactly one aviaroute (can have interchanges).
Major of every city X counted amount of such numberings of all cities from 1 to n , such that on

every aviaroute with the beginning in X, numbers of cities are in ascending order. Every major, except one, noticed that results of counting are multiple of 2016.
Prove, that result of last major is multiple of 2016 too.

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7 All russian olympiad 2016,Day 2 ,grade 9,P8 :
Let a, b, c, d be are positive numbers such that $a+b+c+d=3$ .Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\le\frac{1}{a^2b^2c^2d^2}$$

All russian olympiad 2016,Day 2,grade 11,P7 :
Let a, b, c, d be are positive numbers such that $a+b+c+d=3$ .Prove that

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}\le\frac{1}{a^3b^3c^3d^3}$$

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8 Medians AM_A, BM_B, CM_C of triangle ABC intersect at M.Let Ω_A be circumcircle of triangle passes through midpoint of AM and tangent to BC at M_A.Define Ω_B and Ω_C analogusly.Prove that Ω_A, Ω_B and Ω_C intersect at one point.(A.Yakubov)
sorry for my mistake in translation :blush: :whistling: .thank you jred for your help :coolspeak:
