Second Round Olympiad 2015

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– (A)

1 Let $a_1,a_2,\dots,a_n$ be real numbers.Prove that you can select $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n\in\{-1,1\}$ such that

$$\left(\sum_{i=1}^n a_i\right)^2+\left(\sum_{i=1}^n \varepsilon_i a_i\right)^2\le (n+1)\left(\sum_{i=1}^n a_i^2\right).$$

2 Let $S=\{A_1,A_2,\dots,A_n\}$, where $A_1,A_2,\dots,A_n$ are n pairwise distinct finite sets $(n\ge 2)$, such that for any $A_i,A_j\in S$, $A_i\cup A_j\in S$. If $k=\min_{1\le i\le n}|A_i|\ge 2$, prove that there exist $x\in\bigcup_{i=1}^n A_i$, such that x is in at least $\frac{n}{k}$ of the sets $A_1,A_2,\dots,A_n$ (Here $|X|$ denotes the number of elements in finite set X).

3 P is a point on arc $\overset{\frown}{BC}$ of the circumcircle of $\triangle ABC$ not containing A, K lies on segment AP such that BK bisects $\angle ABC$. The circumcircle of $\triangle KPC$ meets AC, BD at D, E respectively. PE meets AB at F. Prove that $\angle ABC=2\angle FCB$.

4 Find all positive integers k such that for any positive integer n, $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$.

– (B)

1 Let a,b,c be nonnegative real numbers.Prove that

$$\frac{(a-bc)^2+(b-ca)^2+(c-ab)^2}{(a-b)^2+(b-c)^2+(c-a)^2}\ge\frac{1}{2}.$$

2 In isoceles $\triangle ABC$, $AB=AC$, I is its incenter, D is a point inside $\triangle ABC$ such that I,B,C,D are concyclic. The line through C parallel to BD meets AD at E. Prove that $CD^2=BD\cdot CE$.

3 Prove that there exist infinitely many positive integer triples $(a,b,c)(a,b,c>2015)$ such that

$$a|bc-1, b|ac+1, c|ab+1.$$

4 Given positive integers $m,n(2\leq m\leq n)$, let $a_1,a_2,\ldots,a_m$ be a permutation of any m pairwise distinct numbers taken from $1,2,\ldots,n$. If there exist $k\in\{1,2,\ldots,m\}$ such that a_k+k is odd, or there exist positive integers $k,l(1\leq k<l\leq m)$ such that $a_k>a_l$, then call $a_1,a_2,\ldots,a_m$ a *good* sequence. Find the number of good sequences.