KJMO 2019

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–	day 1
1	Each integer coordinates are colored with one color and at least 5 colors are used to color every integer coordinates. Two integer coordinates (x, y) and (z, w) are colored in the same color if $x - z$ and $y - w$ are both multiples of 3. Prove that there exists a line that passes through exactly three points when five points with different colors are chosen randomly.
2	In an acute triangle ABC, point D is on the segment AC such that $\overline{AD} = \overline{BC}$ and $\overline{AC}^2 - \overline{AD}^2 = \overline{AC} \cdot \overline{AD}$. The line that is parallel to the bisector of $\angle ACB$ and passes the point D meets the segment AB at point E. Prove, if $\overline{AE} = \overline{CD}$, $\angle ADB = 3\angle BAC$.
3	Find all pairs of prime numbers $p,\ q (p \le q)$ satisfying the following condition: There exists a natural number n such that $2^n + 3^n + \cdots + (2pq - 1)^n$ is a multiple of $2pq$.
4	$\{a_n\}$ is a sequence of natural numbers satisfying the following inequality for all natural number n: $$(a_1 + \cdots + a_n) \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) \le n^2 + 2019$$ Prove that $\{a_n\}$ is constant.
–	day 2
5	For prime number p, prove that there are integers a, b, c, d such that for every integer n, the expression $n^4 + 1 - \left(n^2 + an + b\right)\left(n^2 + cn + d\right)$ is a multiple of p.
6	Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfies the followings. (Note that $\mathbb{R}$ stands for the set of all real numbers) (1) For each real numbers x, y, the equality $f(x + f(x) + xy) = 2f(x) + xf(y)$ holds. (2) For every real number z, there exists x such that $f(x) = z$.
7	Let O be the circumcenter of an acute triangle ABC. Let D be the intersection of the bisector of the angle A and BC. Suppose that $\angle ODC = 2\angle DAO$. The circumcircle of ABD meets the line segment OA and the line OD at $E(\neq A, O)$, and $F(\neq D)$, respectively. Let X be the intersection of the line DE and the line segment AC. Let Y be the intersection of the bisector of the angle BAF and the segment BE. Prove that $\frac{\overline{AY}}{\overline{BY}} = \frac{\overline{EX}}{\overline{EO}}$.

8 There are two airlines A and B and finitely many airports. For each pair of airports, there is exactly one airline among A and B whose flights operates in both directions. Each airline plans to develop world travel packages which pass each airport exactly once using only its flights. Let a and b be the number of possible packages which belongs to A and B respectively. Prove that $a-b$ is a multiple of 4.

The official statement of the problem has been changed. The above is the form which appeared during the contest. Now the condition 'the number of airports is no less than 4'is attached. Cite the following link.
`https://artofproblemsolving.com/community/c6h2923697p26140823`