Problems from 2021 Thailand Mathematical Olympiad

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– Day 1

1 Let $\triangle ABC$ be an isosceles triangle such that $AB = AC$. Let ω be a circle centered at A with a radius strictly less than AB. Draw a tangent from B to ω at P, and draw a tangent from C to ω at Q. Suppose that the line PQ intersects the line BC at point M. Prove that M is the midpoint of BC.

2 Determine all sequences $a_1, a_2, a_3, \dots$ of positive integers that satisfy the equation

$$(n^2+1)a_{n+1} - a_n = n^3 + n^2 + 1$$

for all positive integers n.

3 Let a, b, and c be positive real numbers satisfying $ab + bc + ca = abc$. Determine the minimum value of

$$a^a bc + b^b ca + c^c ab.$$

4 Kan Krao Park is a circular park that has 21 entrances and a straight line walkway joining each pair of two entrances. No three walkways meet at a single point. Some walkways are paved with bricks, while others are paved with asphalt.

At each intersection of two walkways, excluding the entrances, is planted lotus if the two walkways are paved with the same material, and is planted waterlily if the two walkways are paved with different materials.

Each walkway is decorated with lights if and only if the same type of plant is placed at least 45 different points along that walkway. Prove that there are at least 11 walkways decorated with lights and paved with the same material.

5 Determine all triples (p, m, k) of positive integers such that p is a prime number, m and k are odd integers, and $m^4 + 4^k p^4$ divides $m^2(m^4 - 4^k p^4)$.

– Day 2

6 The cheering team of Ubon Ratchathani University sits on the amphitheater that has 441 seats arranged into a 21×21 grid. Every seat is occupied by exactly one person, and each person has a blue sign and a yellow sign.

Count the number of ways for each person to raise one sign so that each row and column has an odd number of people raising a blue sign.

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7 Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the equation

$$f(xy) = f(x)f(y)f(x+y)$$

for all positive real numbers x and y.

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8 Let P be a point inside an acute triangle ABC. Let the lines BP and CP intersect the sides AC and AB at D and E, respectively. Let the circles with diameters BD and CE intersect at points S and T. Prove that if the points A, S, and T are colinear, then P lies on a median of $\triangle ABC$.

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9 Let S be a set of positive integers such that if a and b are elements of S such that $a < b$, then $b - a$ divides the least common multiple of a and b, and the quotient is an element of S. Prove that the cardinality of S is less than or equal to 2.

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10 Let $d \geq 13$ be an integer, and let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ be a polynomial of degree d with complex coefficients such that $a_n = a_{d-n}$ for all $n \in \{0, 1, \ldots, d\}$. Prove that if P has no double roots, then P has two distinct roots z_1 and z_2 such that $|z_1 - z_2| < 1$.
