Berkeley Math Tournament , November 20 & 21, 2021 , Geometry, Discrete, Guts, General, Algebra Round

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–	Discrete Round
1	Towa has a hand of three different red cards and three different black cards. How many ways can Towa pick a set of three cards from her hand that uses at least one card of each color?
2	Alice is counting up by fives, starting with the number 3. Meanwhile, Bob is counting down by fours, starting with the number 2021. How many numbers between 3 and 2021, inclusive, are counted by both Alice and Bob?
3	How many distinct sums can be made from adding together exactly 8 numbers that are chosen from the set $\{1, 4, 7, 10\}$, where each number in the set is chosen at least once? (For example, one possible sum is $1 + 1 + 1 + 4 + 7 + 7 + 10 + 10 = 41$.)
4	Derek and Julia are two of 64 players at a casual basketball tournament. The players split up into 8 teams of 8 players at random. Each team then randomly selects 2 captains among their players. What is the probability that both Derek and Julia are captains?
5	How many three-digit numbers $\underline{abc}$ have the property that when it is added to $\underline{cba}$, the number obtained by reversing its digits, the result is a palindrome? (Note that $\underline{cba}$ is not necessarily a three-digit number since before reversing, c may be equal to 0.)
6	Compute the sum of all positive integers n such that n^n has 325 positive integer divisors. (For example, $4^4 = 256$ has 9 positive integer divisors: 1, 2, 4, 8, 16, 32, 64, 128, 256.)
7	For a given positive integer n, you may perform a series of steps. At each step, you may apply an operation: you may increase your number by one, or if your number is divisible by 2, you may divide your number by 2. Let $\ell(n)$ be the minimum number of operations needed to transform the number n to 1 (for example, $\ell(1) = 0$ and $\ell(7) = 4$). How many positive integers n are there such that $\ell(n) \le 12$?
8	Consider the randomly generated base 10 real number $r = 0.\overline{p_0p_1p_2\ldots}$, where each p_i is a digit from 0 to 9, inclusive, generated as follows: p_0 is generated uniformly at random from 0 to 9, inclusive, and for all $i \ge 0$, p_{i+1} is generated uniformly at random from p_i to 9, inclusive. Compute the expected value of r.
9	Let $p = 101$. The sum

$$\sum_{k=1}^{10} \frac{1}{\binom{p}{k}}$$

can be written as a fraction of the form $\frac{a}{p!}$, where a is a positive integer. Compute $a \pmod p$.

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10 Let N be the number of ways to draw 22 straight edges between 10 labeled points, of which no three are collinear, such that no triangle with vertices among these 10 points is created, and there is at most one edge between any two labeled points. Compute $\frac{N}{9!}$.

T1 How many integers n from 1 to 2020, inclusive, are there such that 2020 divides $n^2 + 1$?

T2 A gradian is a unit of measurement of angles much like degrees, except that there are 100 gradians in a right angle. Suppose that the number of gradians in an interior angle of a regular polygon with m sides equals the number of degrees in an interior angle of a regular polygon with n sides. Compute the number of possible distinct ordered pairs (m, n).

T3 Let N be the number of tuples $(a_1, a_2, ..., a_{150})$ satisfying: • $a_i \in \{2, 3, 5, 7, 11\}$ for all $1 \le i \le 99$. • $a_i \in \{2, 4, 6, 8\}$ for all $100 \le i \le 150$. • $\sum_{i=1}^{150} a_i$ is divisible by 8.
Compute the last three digits of N.

– Guts Round

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1 The isoelectric point of glycine is the pH at which it has zero charge. Its charge is $-\frac{1}{3}$ at pH 3.55, while its charge is $\frac{1}{2}$ at pH 9.6. Charge increases linearly with pH. What is the isoelectric point of glycine?

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2 The battery life on a computer decreases at a rate proportional to the display brightness. Austin starts off his day with both his battery life and brightness at 100%. Whenever his battery life (expressed as a percentage) reaches a multiple of 25, he also decreases the brightness of his display to that multiple of 25. If left at 100% brightness, the computer runs out of battery in 1 hour. Compute the amount of time, in minutes, it takes for Austin's computer to reach 0% battery using his modified scheme.

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3 Compute $\log_2 6 \cdot \log_3 72 - \log_2 9 - \log_3 8$.

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4 Compute the sum of all real solutions to $4^x - 2021 \cdot 2^x + 1024 = 0$.

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5 Anthony the ant is at point A of regular tetrahedron $ABCD$ with side length 4. Anthony wishes to crawl on the surface of the tetrahedron to the midpoint of $\overline{BC}$. However, he does not want to touch the interior of face $\triangle ABC$, since it is covered with lava. What is the shortest distance Anthony must travel?

6 Three distinct integers are chosen uniformly at random from the set

$$\{2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030\}.$$

Compute the probability that their arithmetic mean is an integer.

7 Ditty can bench 80 pounds today. Every week, the amount he benches increases by the largest prime factor of the weight he benched in the previous week. For example, since he started benching 80 pounds, next week he would bench 85 pounds. What is the minimum number of weeks from today it takes for Ditty to bench at least 2021 pounds?

8 Let $\overline{AB}$ be a line segment with length 10. Let P be a point on this segment with $AP = 2$. Let ω_1 and ω_2 be the circles with diameters $\overline{AP}$ and $\overline{PB}$, respectively. Let XY be a line externally tangent to ω_1 and ω_2 at distinct points X and Y , respectively. Compute the area of $\triangle XPY$.

9 Druv has a 33×33 grid of unit squares, and he wants to color each unit square with exactly one of three distinct colors such that he uses all three colors and the number of unit squares with each color is the same. However, he realizes that there are internal sides, or unit line segments that have exactly one unit square on each side, with these two unit squares having different colors. What is the minimum possible number of such internal sides?

10 Compute the number of nonempty subsets S of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ such that $\frac{\max\ S + \min\ S}{2}$ is an element of S.

11 Compute the sum of all prime numbers p with $p \geq 5$ such that p divides $(p + 3)^{p-3} + (p + 5)^{p-5}$.
.

12 Unit square $ABCD$ is drawn on a plane. Point O is drawn outside of $ABCD$ such that lines AO and BO are perpendicular. Square $FROG$ is drawn with F on AB such that $AF = \frac{2}{3}$, R is on $\overline{BO}$, and G is on $\overline{AO}$. Extend segment $\overline{OF}$ past $\overline{AB}$ to intersect side $\overline{CD}$ at E. Compute DE.

13 How many ways are there to completely fill a 3×3 grid of unit squares with the letters B, M, and T, assigning exactly one of the three letters to each of the squares, such that no 2 adjacent unit squares contain the same letter? Two unit squares are adjacent if they share a side.

14 Given an integer c, the sequence $a_0, a_1, a_2, ...$ is generated using the recurrence relation $a_0 = c$ and $a_i = a_{i-1}^i + 2021a_{i-1}$ for all $i \geq 1$. Given that $a_0 = c$, let $f(c)$ be the smallest positive integer n such that $a_n - 1$ is a multiple of 47. Compute

$$\sum_{k=1}^{46} f(k).$$

15 Compute

$$\frac{\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{24}\right)\cos\left(\frac{\pi}{48}\right)\cos\left(\frac{\pi}{96}\right)...}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{32}\right)...}$$

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16 Sigfried is singing the ABC's 100 times straight, for some reason. It takes him 20 seconds to sing the ABC's once, and he takes a 5 second break in between songs. Normally, he sings the ABC's without messing up, but he gets fatigued when singing correctly repeatedly. For any song, if he sung the previous three songs without messing up, he has a $\frac{1}{2}$ chance of messing up and taking 30 seconds for the song instead. What is the expected number of minutes it takes for Sigfried to sing the ABC's 100 times? Round your answer to the nearest minute.

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17 Triangle $\triangle ABC$ has circumcenter O and orthocenter H. Let D be the foot of the altitude from A to BC, and suppose $AD = 12$. If $BD = \frac{1}{4}BC$ and $OH \parallel BC$, compute AB^2.
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18 The equation $\sqrt[3]{\sqrt[3]{x-\frac{3}{8}}-\frac{3}{8}} = x^3+\frac{3}{8}$ has exactly two real positive solutions r and s. Compute $r+s$.

19-21 Guts Round / Set 7

p19. Let a be the answer to Problem 19, b be the answer to Problem 20, and c be the answer to Problem 21.

Compute the real value of a such that

$$\sqrt{a(101b+1)}-1 = \sqrt{b(c-1)}+10\sqrt{(a-c)b}.$$

p20. Let a be the answer to Problem 19, b be the answer to Problem 20, and c be the answer to Problem 21.

For some triangle $\triangle\ ABC$, let ω and ω_A be the incircle and A-excircle with centers I and I_A, respectively. Suppose AC is tangent to ω and ω_A at E and E', respectively, and AB is tangent to ω and ω_A at F and F' respectively. Furthermore, let P and Q be the intersections of BI with EF and CI with EF, respectively, and let P' and Q' be the intersections of BI_A with $E'F'$ and CI_A with $E'F'$, respectively. Given that the circumradius of $\triangle\ ABC$ is a, compute the maximum integer value of BC such that the area $[PQP'Q']$ is less than or equal to 1.

p21. Let a be the answer to Problem 19, b be the answer to Problem 20, and c be the answer to Problem 21.

Let c be a positive integer such that $gcd(b, c) = 1$. From each ordered pair (x, y) such that x and y are both integers, we draw two lines through that point in the $x-y$ plane, one with slope $\frac{b}{c}$ and one with slope $-\frac{c}{b}$. Given that the number of intersections of these lines in $[0, 1)^2$ is a square

number, what is the smallest possible value of c?
Note that $[0,1)^2$ refers to all points (x,y) such that $0 \leq x < 1$ and $0 \leq y < 1$.

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22 In $\triangle ABC$, let D and E be points on the angle bisector of $\angle BAC$ such that $\angle ABD = \angle ACE = 90^o$. Furthermore, let F be the intersection of AE and BC, and let O be the circumcenter of $\triangle AFC$. If $\frac{AB}{AC} = \frac{3}{4}$, $AE = 40$, and BD bisects EF, compute the perpendicular distance from A to OF.

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23 Alireza is currently standing at the point $(0,0)$ in the $x-y$ plane. At any given time, Alireza can move from the point (x,y) to the point $(x+1,y)$ or the point $(x,y+1)$. However, he cannot move to any point of the form (x,y) where $y \equiv 2x \pmod 5$. Let p_k be the number of paths Alireza can take starting from the point $(0,0)$ to the point $(k+1, 2k+1)$. Evaluate the sum

$$\sum_{k=1}^{\infty} \frac{p_k}{5^k}.$$

.

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24 Suppose that a, b, c, and p are positive integers such that p is a prime number and

$$a^2 + b^2 + c^2 = ab + bc + ca + 2021p$$

.
Compute the least possible value of $\max(a, b, c)$.

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25 For any $p, q \in N$, we can express $\frac{p}{q}$ as the base 10 decimal $x_1x_2...x_\ell.x_{\ell+1}...x_a\overline{y_1y_2...y_b}$, with the digits $y_1, ...y_b$ repeating. In other words, $\frac{p}{q}$ can be expressed with integer part $x_1x_2...x_\ell$ and decimal part $0.x_{\ell+1}...x_a\overline{y_1y_2...y_b}$. Given that $\frac{p}{q} = \frac{(2021)^{2021}}{2021!}$, estimate the minimum value of a. If E is the exact answer to this question and A is your answer, your score is given by $\max\left(0, \left\lfloor 25 - \frac{1}{10}|E-A|\right\rfloor\right)$.

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26 Kailey starts with the number 0, and she has a fair coin with sides labeled 1 and 2. She repeatedly flips the coin, and adds the result to her number. She stops when her number is a positive perfect square. What is the expected value of Kailey's number when she stops? If E is your estimate and A is the correct answer, you will receive $\left\lfloor 25e^{-\frac{5|E-A|}{2}} \right\rfloor$ points.

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27 Let $S = 1, 2, 2^2, 2^3, ..., 2^{2021}$. Compute the difference between the number of even digits and the number of odd digits across all numbers in S (written as integers in base 10 with no leading zeros). If E is the exact answer to this question and A is your answer, your score is given by $\max\left(0, \left\lfloor 25 - \frac{1}{2\cdot 10^8}|E-A|^4 \right\rfloor\right)$.

– Geometry Round

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1 Shreyas has a rectangular piece of paper $ABCD$ such that $AB = 20$ and $AD = 21$. Given that Shreyas can make exactly one straight-line cut to split the paper into two pieces, compute the maximum total perimeter of the two pieces

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2 Compute the area of the smallest triangle which can contain six congruent, non-overlapping unit circles.

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3 In quadrilateral $ABCD$, suppose that $\overline{CD}$ is perpendicular to $\overline{BC}$ and $\overline{DA}$. Point E is chosen on segment $\overline{CD}$ such that $\angle AED = \angle BEC$. If $AB = 6$, $AD = 7$, and $\angle ABC = 120^o$, compute $AE + EB$.

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4 An equilateral polygon has unit side length and alternating interior angle measures of 15^o and 300^o. Compute the area of this polygon.

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5 Let circles ω_1 and ω_2 intersect at P and Q. Let the line externally tangent to both circles that is closer to Q touch ω_1 at A and ω_2 at B. Let point T lie on segmentPQ such that $\angle ATB = 90^o$. Given that $AT = 6$, $BT = 8$, and $PT = 4$, compute PQ.

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6 Consider 27 unit-cubes assembled into one $3 \times 3 \times 3$ cube. Let A and B be two opposite corners of this large cube. Remove the one unit-cube not visible from the exterior, along with all six unit-cubes in the center of each face. Compute the minimum distance an ant has to walk along the surface of the modified cube to get from A to B.
https://cdn.artofproblemsolving.com/attachments/0/5/d3aa802eae40cfe717088445daabd5e71946
png

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7 The line ℓ passes through vertex$B$ and the interior of regular hexagon $ABCDEF$. If the distances from ℓ to the vertices A and C are 7 and 4, respectively, compute the area of hexagon $ABCDEF$.

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8 Let $\triangle ABC$ be a triangle with $AB = 15$, $AC = 13$, $BC = 14$, and circumcenter O. Let ℓ be the line through A perpendicular to segment BC. Let the circumcircle of $\triangle AOB$ and the circumcircle of $\triangle AOC$ intersect ℓ at points X and Y (other than A), respectively. Compute the length of $\overline{XY}$
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9 Let $ABCD$ be a convex quadrilateral such that $\triangle ABC$ is equilateral. Let P be a point inside the quadrilateral such that $\triangle APD$ is equilateral and $\angle PCD = 30^o$. Given that $CP = 2$ and $CD = 3$, compute the area of the triangle formed by P, the midpoint of segment $\overline{BC}$, and the midpoint of segment $\overline{AB}$.

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10 Consider $\triangle ABC$ such that $CA + AB = 3BC$. Let the incircle ω touch segments $\overline{CA}$ and $\overline{AB}$ at E and F, respectively, and define P and Q such that segments $\overline{PE}$ and $\overline{QF}$ are diameters of ω. Define the function D of a point K to be the sum of the distances from K to P and Q (i.e. $D(K) = KP + KQ$). Let W, X, Y , and Z be points chosen on lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CE}$, $\overleftrightarrow{EF}$, and $\overleftrightarrow{FB}$, respectively. Given that $BC = \sqrt{133}$ and the inradius of $\triangle ABC$ is $\sqrt{14}$, compute the minimum value of $D(W) + D(X) + D(Y) + D(Z)$.

Tie 1 Regular hexagon $NOSAME$ with side length 1 and square $UDON$ are drawn in the plane such that $UDON$ lies outside of $NOSAME$. Compute $[SAND] + [SEND]$, the sum of the areas of quadrilaterals $SAND$ and $SEND$.

Tie 2 Let $\triangle A_0B_0C_0$ be an equilateral triangle with area 1, and let A_1, B_1, C_1 be the midpoints of $\overline{A_0B_0}$, $\overline{B_0C_0}$, and $\overline{C_0A_0}$, respectively. Furthermore, set A_2, B_2, C_2 as the midpoints of segments $\overline{A_0A_1}$, $\overline{B_0B_1}$, and $\overline{C_0C_1}$ respectively. For $n \geq 1$, A_{2n+1} is recursively defined as the midpoint of $A_{2n}A_{2n-1}$, and A_{2n+2} is recursively defined as the midpoint of $\overline{A_{2n+1}A_{2n-1}}$. Recursively define B_n and C_n the same way. Compute the value of $\lim_{n\to\infty}[A_nB_nC_n]$, where $[A_nB_nC_n]$ denotes the area of triangle $\triangle A_nB_nC_n$.

Tie 3 Right triangle $\triangle ABC$ with its right angle at B has angle bisector $\overline{AD}$ with D on $\overline{BC}$, as well as altitude $\overline{BE}$ with E on $\overline{AC}$. If $\overline{DE} \perp \overline{BC}$ and $AB = 1$, compute AC.

– General Round

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1 Carson and Emily attend different schools. Emily's school has four times as many students as Carson's school. The total number of students in both schools combined is 10105. How many students go to Carson's school?

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2 same as Algebra#1

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3 A scalene acute triangle has angles whose measures (in degrees) are whole numbers. What is the smallest possible measure of one of the angles, in degrees?

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4 Moor and Samantha are drinking tea at a constant rate. If Moor starts drinking tea at $8:00$ am, he will finish drinking 7 cups of tea by $12:00$ pm. If Samantha joins Moor at $10:00$ am, they will finish drinking the 7 cups of tea by $11:15$ am. How many hours would it take Samantha to drink 1 cup of tea?

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5 Bill divides a 28×30 rectangular board into two smaller rectangular boards with a single straight-cut, so that the side lengths of both boards are positive whole numbers. How many different pairs of rectangular boards, up to congruence and arrangement, can Bill possibly obtain? (For instance, a cut that is 1 unit away from either of the edges with length 28 will result in the same pair of boards: either way, one would end up with a 1×28 board and a 29×28 board.)

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6 A toilet paper roll is a cylinder of radius 8 and height 6 with a hole in the shape of a cylinder of radius 2 and the same height. That is, the bases of the roll are annuli with inner radius 2 and outer radius 8. Compute the surface area of the roll.

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7 Alice is counting up by fives, starting with the number 3. Meanwhile, Bob is counting down by fours, starting with the number 2021. How many numbers between 3 and 2021, inclusive, are counted by both Alice and Bob?

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8 On the first day of school, Ashley the teacher asked some of her students what their favorite color was and used those results to construct the pie chart pictured below. During this first day, 165 students chose yellow as their favorite color. The next day, she polled 30 additional students and was shocked when none of them chose yellow. After making a new pie chart based on the combined results of both days, Ashley noticed that the angle measure of the sector representing the students whose favorite color was yellow had decreased. Compute the difference, in degrees, between the old and the new angle measures.
`https://cdn.artofproblemsolving.com/attachments/2/5/f605bf8d684075fe13fee9eb44f8f50b64c7c png`

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9 Rakesh is flipping a fair coin repeatedly. If T denotes the event where the coin lands on tails and H denotes the event where the coin lands on heads, what is the probability Rakesh flips the sequence HHH before the sequence THH?

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10 Triangle $\triangle ABC$ has side lengths $AB = AC = 27$ and $BC = 18$. Point D is on $\overline{AB}$ and point E is on $\overline{AC}$ such that $\angle BCD = \angle CBE = \angle BAC$. Compute DE.

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11 Compute the number of sequences of five positive integers $a_1, ..., a_5$ where all $a_i \le 5$ and the greatest common divisor of all five integers is 1.

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12 Let a, b, and c be the solutions of the equation

$$x^3 - 3 \cdot 2021^2 x = 2 \cdot 20213.$$

Compute $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

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13 A six-sided die is rolled four times. What is the probability that the minimum value of the four rolls is 4?

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14 Let $r_1, r_2, ..., r_{47}$ be the roots of $x^{47} - 1 = 0$. Compute

$$\sum_{i=1}^{47} r_i^{2020}.$$

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15 Benji has a 2×2 grid, which he proceeds to place chips on. One by one, he places a chip on one of the unit squares of the grid at random. However, if at any point there is more than one chip on the same square, Benji moves two chips on that square to the two adjacent squares, which he calls a chip-fire. He keeps adding chips until there is an infinite loop of chip-fires. What is the expected number of chips that will be added to the board?

16 Jason and Valerie agree to meet for game night, which runs from $4:00$ PM to $5:00$ PM. Jason and Valerie each choose a random time from $4:00$ PM to $5:00$ PM to show up. If Jason arrives first, he will wait 20 minutes for Valerie before leaving. If Valerie arrives first, she will wait 10 minutes for Jason before leaving. What is the probability that Jason and Valerie successfully meet each other for game night?

17 Simplify $\sqrt[4]{17+12\sqrt{2}}-\sqrt[4]{17-12\sqrt{2}}$.

18 same as Geometry#3

19 same as Discrete#5

20 For some positive integer n, $(1+i)+(1+i)^2+(1+i)^3+...+(1+i)^n=(n^2-1)(1-i)$, where $i=\sqrt{-1}$. Compute the value of n.

21 There exist integers a and b such that $(1+\sqrt{2})^{12}=a+b\sqrt{2}$. Compute the remainder when ab is divided by 13.

22 Austin is at the Lincoln Airport. He wants to take 5 successive flights whose destinations are randomly chosen among Indianapolis, Jackson, Kansas City, Lincoln, and Milwaukee. The origin and destination of each flight may not be the same city, but Austin must arrive back at Lincoln on the last of his flights. Compute the probability that the cities Austin arrives at are all distinct.

23 Shivani has a single square with vertices labeled $ABCD$. She is able to perform the following transformations: • She does nothing to the square. • She rotates the square by 90, 180, or 270 degrees. • She reflects the square over one of its four lines of symmetry.
For the first three timesteps, Shivani only performs reflections or does nothing. Then for the next three timesteps, she only performs rotations or does nothing. She ends up back in the square's original configuration. Compute the number of distinct ways she could have achieved this.

24 Given that x, y, and z are a combination of positive integers such that $xyz=2(x+y+z)$, compute the sum of all possible values of $x+y+z$.

25 Let $\triangle BMT$ be a triangle with $BT=1$ and height 1. Let O_0 be the centroid of $\triangle BMT$, and let $\overline{BO_0}$ and $\overline{TO_0}$ intersect $\overline{MT}$ and $\overline{BM}$ at B_1 and T_1, respectively. Similarly, let O_1 be the centroid of $\triangle B_1MT_1$, and in the same way, denote the centroid of $\triangle B_nMT_n$ by O_n, the intersection of $\overline{BO_n}$ with $\overline{MT}$ by B_{n+1}, and the intersection of $\overline{TO_n}$ with $\overline{BM}$ by T_{n+1}. Compute the area of quadrilateral $MBO_{2021}T$.

T1 The arithmetic mean of $2, 6, 8$, and x is 7. The arithmetic mean of $2, 6, 8, x$, and y is 9. What is the value of $y-x$?

T2	Compute the radius of the largest circle that fits entirely within a unit cube.
T3	Dexter and Raquel are playing a game with N stones. Dexter goes first and takes one stone from the pile. After that, the players alternate turns and can take anywhere from 1 to $x+1$ stones from the pile, where x is the number of stones the other player took on the turn immediately prior. The winner is the one to take the last stone from the pile. Assuming Dexter and Raquel play optimally, compute the number of positive integers $N \leq 2021$ where Dexter wins this game.
T4	Let z_1, z_2, and z_3 be the complex roots of the equation $(2z - 3\overline{z})^3 = 54i + 54$. Compute the area of the triangle formed by z_1, z_2, and z_3 when plotted in the complex plane.
T5	Let r, s, t, u be the distinct roots of the polynomial $x^4 + 2x^3 + 3x^2 + 3x + 5$. For $n \geq 1$, define $s_n = r^n + s^n + t^n + u^n$ and $t_n = s_1 + s_2 + ... + s_n$. Compute $t_4 + 2t_3 + 3t_2 + 3t_1 + 5$.
–	Algebra Round
1	Let x be a real number such that $x^2 - x + 1 = 7$ and $x^2 + x + 1 = 13$. Compute the value of x^4.
2	Let f and g be linear functions such that $f(g(2021)) - g(f(2021)) = 20$. Compute $f(g(2022)) - g(f(2022))$. (Note: A function h is linear if $h(x) = ax + b$ for all real numbers x.)
3	Let x be a solution to the equation $\lfloor x \lfloor x + 2 \rfloor + 2 \rfloor = 10$. Compute the smallest C such that for any solution x, $x < C$. Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor -4.25 \rfloor = -5$.
4	Let θ be a real number such that $1 + \sin 2\theta - \left(\frac{1}{2}\sin 2\theta\right)^2 = 0$. Compute the maximum value of $(1 + \sin\theta)(1 + \cos\theta)$.
5	Compute the sum of the real solutions to $\lfloor x \rfloor \{x\} = 2020x$. Here, $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x, and$\{x\} = x - \lfloor x \rfloor$.
6	Let f be a real function such that for all $x \neq 0$, $x \neq 1$, $$f(x) + f\left(-\frac{1}{x-1}\right) = \frac{9}{4x^2} + f\left(1 - \frac{1}{x}\right).$$ Compute $f\left(\frac{1}{2}\right)$. .

7 Let $z_1, z_2, ..., z_{2020}$ be the roots of the polynomial $z^{2020} + z^{2019} + ... + z + 1$. Compute

$$\sum_{i=1}^{2020} \frac{1}{1 - z_i^{2020}}.$$

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8 Let $f(w) = w^3 - rw^2 + sw - \frac{4\sqrt{2}}{27}$ denote a polynomial, where $r^2 = \left(\frac{8\sqrt{2}+10}{7}\right) s$. The roots of f correspond to the sides of a right triangle. Compute the smallest possible area of this triangle

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9 Compute the sum of the positive integers $n \leq 100$ for which the polynomial $x^n + x + 1$ can be written as the product of at least 2 polynomials of positive degree with integer coefficients.

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10 Given a positive integer n, define $f_n(x)$ to be the number of square-free positive integers k such that $kx \leq n$. Then, define $v_(n)$ as

$$v(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_n(i^2) - 6f_n(ij) + f_n(j^2).$$

Compute the largest positive integer $2 \leq n \leq 100$ for which $v(n) - v(n-1)$ is negative.
(Note: A square-free positive integer is a positive integer that is not divisible by the square of any prime.)

Tie 1 Let the sequence $\{a_n\}$ for $n \geq 0$ be defined as $a_0 = c$, and for $n \geq 0$,

$$a_n = \frac{2a_{n-1}}{4a_{n-1}^2 - 1}.$$

Compute the sum of all values of c such that a_{2020} exists but a_{2021} does not exist.

Tie 2 Real numbers x and y satisfy the equations $x^2 - 12y = 17^2$ and $38x - y^2 = 2 \cdot 7^3$. Compute $x + y$.

Tie 3 For integers a and b, $a + b$ is a root of $x^2 + ax + b = 0$. Compute the smallest possible value of ab.
