Cono Sur Olympiad 2021

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–	Day 1
1	We say that a positive integer is guarani if the sum of the number with its reverse is a number that only has odd digits. For example, 249 and 30 are guarani, since $249+942 = 1191$ and $30+03=33$. a) How many 2021-digit numbers are guarani? b) How many 2023-digit numbers are guarani?
2	Let ABC be a triangle and I its incenter. The lines BI and CI intersect the circumcircle of ABC again at M and N, respectively. Let C_1 and C_2 be the circumferences of diameters NI and MI, respectively. The circle C_1 intersects AB at P and Q, and the circle C_2 intersects AC at R and S. Show that P, Q, R and S are concyclic.
3	In a tennis club, each member has exactly $k>0$ friends, and a tournament is organized in rounds such that each pair of friends faces each other in matches exactly once. Rounds are played in simultaneous matches, choosing pairs until they cannot choose any more (that is, among the unchosen people, there is not a pair of friends which has its match pending). Determine the maximum number of rounds the tournament can have, depending on k.
–	Day 2
4	In a heap there are 2021 stones. Two players A and B play removing stones of the pile, alternately starting with A. A valid move for A consists of remove $1,2$ or 7 stones. A valid move for B is to remove $1,3,4$ or 6 stones. The player who leaves the pile empty after making a valid move wins. Determine if some of the players have a winning strategy. If such a strategy exists, explain it.
5	Given an integer $n\geq 3$, determine if there are n integers $b_1,b_2,\dots,b_n$, distinct two-by-two (that is, $b_i\neq b_j$ for all $i\neq j$) and a polynomial $P(x)$ with coefficients integers, such that $P(b_1)=b_2,P(b_2)=b_3,\dots,P(b_{n-1})=b_n$ and $P(b_n)=b_1$.
6	Let ABC be a scalene triangle with circle Γ. Let P,Q,R,S distinct points on the BC side, in that order, such that $\angle BAP = \angle CAS$ and $\angle BAQ = \angle CAR$. Let U,V,W,Z be the intersections, distinct from A, of the AP,AQ,AR and AS with Γ, respectively. Let $X=UQ\cap SW, Y=PV\cap ZR$, $T=UR\cap VS$ and $K=PW\cap ZQ$. Suppose that the points M and N are well determined, such that $M=KX\cap TY$ and $N=TX\cap KY$. Show that M,N,A are collinear.