

**ITAMO 2016**

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by randomusername

- Cesenatico, May 6, 2016  
Time allowed: 4 hours and 30 minutes

**1** Let  $ABC$  be a triangle, and let  $D$  and  $E$  be the orthogonal projections of  $A$  onto the internal bisectors from  $B$  and  $C$ . Prove that  $DE$  is parallel to  $BC$ .

**2** A mathematical contest had 3 problems, each of which was given a score between 0 and 7 (0 and 7 included). It is known that, for any two contestants, there exists at most one problem in which they have obtained the same score (for example, there are no two contestants whose ordered scores are 7, 1, 2 and 7, 1, 5, but there might be two contestants whose ordered scores are 7, 1, 2 and 7, 2, 1). Find the maximum number of contestants.

**3** Let  $\Gamma$  be the excircle of triangle  $ABC$  opposite to the vertex  $A$  (namely, the circle tangent to  $BC$  and to the prolongations of the sides  $AB$  and  $AC$  from the part  $B$  and  $C$ ). Let  $D$  be the center of  $\Gamma$  and  $E, F$ , respectively, the points in which  $\Gamma$  touches the prolongations of  $AB$  and  $AC$ . Let  $J$  be the intersection between the segments  $BD$  and  $EF$ .

Prove that  $\angle CJB$  is a right angle.

**4** Determine all pairs of positive integers  $(a, n)$  with  $a \geq n \geq 2$  for which  $(a + 1)^n + a - 1$  is a power of 2.

**5** Let  $x_0, x_1, x_2, \dots$  be a sequence of rational numbers defined recursively as follows:  $x_0$  can be any rational number and, for  $n \geq 0$ ,

$$x_{n+1} = \begin{cases} \left\lfloor \frac{x_n}{2} - 1 \right\rfloor & \text{if the numerator of } x_n \text{ is even,} \\ \left\lfloor \frac{1}{x_n} - 1 \right\rfloor & \text{if the numerator of } x_n \text{ is odd,} \end{cases}$$

where by numerator of a rational number we mean the numerator of the fraction in its lowest terms. Prove that for any value of  $x_0$ :

- (a) the sequence contains only finitely many distinct terms;
- (b) the sequence contains exactly one of the numbers 0 and  $2/3$  (namely, either there exists an index  $k$  such that  $x_k = 0$ , or there exists an index  $m$  such that  $x_m = 2/3$ , but not both).

**6** A mysterious machine contains a secret combination of 2016 integer numbers  $x_1, x_2, \dots, x_{2016}$ . It is known that all the numbers in the combination are equal but one. One may ask questions to the machine by giving to it a sequence of 2016 integer numbers  $y_1, \dots, y_{2016}$ , and the machine

answers by telling the value of the sum

$$x_1y_1 + \cdots + x_{2016}y_{2016}.$$

After answering the first question, the machine accepts a second question and then a third one, and so on.

Determine how many questions are necessary to determine the combination:

- (a) knowing that the number which is different from the others is equal to zero;
  - (b) not knowing what the number different from the others is.
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