ICMC 2021-2022

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by mastermind.hk16

Round 1 28 November 2021

1 Let T_n be the number of non-congruent triangles with positive area and integer side lengths summing to n. Prove that $T_{2022} = T_{2019}$.

Proposed by Constantinos Papachristoforou

2 Find all integers n for which there exists a table with n rows, 2022 columns, and integer entries, such that subtracting any two rows entry-wise leaves every remainder modulo 2022.

Proposed by Tony Wang

3 Let $\mathcal{M}$ be the set of $n \times n$ matrices with integer entries. Find all $A \in \mathcal{M}$ such that $\det(A+B) + \det(B)$ is even for all $B \in \mathcal{M}$.

Proposed by Ethan Tan

4 Let p be a prime number. Find all subsets $S \subseteq \mathbb{Z}/p\mathbb{Z}$ such that
1. if $a, b \in S$, then $ab \in S$, and
2. there exists an $r \in S$ such that for all $a \in S$, we have $r - a \in S \cup \{0\}$.

Proposed by Harun Khan

5 A *tanned vector* is a nonzero vector in $\mathbb{R}^3$ with integer entries. Prove that any tanned vector of length at most 2021 is perpendicular to a tanned vector of length at most 100.

Proposed by Ethan Tan

6 Is it possible to cover a circle of area 1 with finitely many equilateral triangles whose areas sum to 1.01, all pointing in the same direction?

Proposed by Ethan Tan

Round 2 27 February 2022

1 Let S be a set of 2022 lines in the plane, no two parallel, no three concurrent. S divides the plane into finite regions and infinite regions. Is it possible for all the finite regions to have integer area?

Proposed by Tony Wang

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2 Evaluate

$$\frac{1/2}{1+\sqrt{2}}+\frac{1/4}{1+\sqrt[4]{2}}+\frac{1/8}{1+\sqrt[8]{2}}+\frac{1/16}{1+\sqrt[16]{2}}+\cdots$$

Proposed by Ethan Tan

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3 A set of points has *point symmetry* if a reflection in some point maps the set to itself. Let $\mathcal{P}$ be a solid convex polyhedron whose orthogonal projections onto any plane have point symmetry. Prove that $\mathcal{P}$ has point symmetry.

Proposed by Ethan Tan

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4 Fix a set of integers S. An integer is *clean* if it is the sum of distinct elements of S in exactly one way, and *dirty* otherwise. Prove that the set of dirty numbers is either empty or infinite.

Note: We consider the empty sum to equal 0.

Proposed by Tony Wang and Ethan Tan

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5 A robot on the number line starts at 1. During the first minute, the robot writes down the number 1. Each minute thereafter, it moves by one, either left or right, with equal probability. It then multiplies the last number it wrote by n/t, where n is the number it just moved to, and t is the number of minutes elapsed. It then writes this number down. For example, if the robot moves right during the second minute, it would write down $2/2=1$.

Find the expected sum of all numbers it writes down, given that it is finite.

Proposed by Ethan Tan

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