International Olympiad of Metropolises 2021

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by parmenides51, rafaello

–	Day 1
1	A positive integer is written on the board. Every minute Maxim adds to the number on the board one of its positive divisors, writes the result on the board and erases the previous number. However, it is forbidden for him to add the same number twice in a row. Prove that he can proceed in such a way that eventually a perfect square will appear on the board.
2	Points P and Q are chosen on the side BC of triangle ABC so that P lies between B and Q. The rays AP and AQ divide the angle BAC into three equal parts. It is known that the triangle APQ is acute-angled. Denote by B_1, P_1, Q_1, C_1 the projections of points B, P, Q, C onto the lines AP, AQ, AP, AQ, respectively. Prove that lines B_1P_1 and C_1Q_1 meet on line BC.
3	Let $a_1, a_2, \ldots, a_n$ ($n \geq 2$) be nonnegative real numbers whose sum is $\frac{n}{2}$. For every $i = 1, \ldots, n$ define $$b_i = a_i + a_ia_{i+1} + a_ia_{i+1}a_{i+2} + \cdots + a_ia_{i+1}\cdots a_{i+n-2} + 2a_ia_{i+1}\cdots a_{i+n-1}$$ where $a_{j+n} = a_j$ for every j. Prove that $b_i \geq 1$ holds for at least one index i.
–	Day 2
4	Six real numbers $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are given. For each triplet of distinct numbers of those six Vitya calculated their sum. It turned out that the 20 sums are pairwise distinct; denote those sums by $$s_1 < s_2 < s_3 < \cdots < s_{19} < s_{20}.$$ It is known that $x_2 + x_3 + x_4 = s_{11}$, $x_2 + x_3 + x_6 = s_{15}$ and $x_1 + x_2 + x_6 = s_m$. Find all possible values of m.
5	There is a safe that can be opened by entering a secret code consisting of n digits, each of them is 0 or 1. Initially, n zeros were entered, and the safe is closed (so, all zeros is not the secret code). In one attempt, you can enter an arbitrary sequence of n digits, each of them is 0 or 1. If the entered sequence matches the secret code, the safe will open. If the entered sequence matches the secret code in more positions than the previously entered sequence, you will hear a click. In any other cases the safe will remain locked and there will be no click. Find the smallest number of attempts that is sufficient to open the safe in all cases.

6 Let $ABCD$ be a tetrahedron and suppose that M is a point inside it such that $\angle MAD = \angle MBC$ and $\angle MDB = \angle MCA$. Prove that

$$MA \cdot MB + MC \cdot MD < \max(AD \cdot BC, AC \cdot BD).$$