Bosnia and Herzegovina Team Selection Test 2016

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Day 1 May 14th

1 Let $ABCD$ be a quadrilateral inscribed in circle k. Lines AB and CD intersect at point E such that $AB = BE$. Let F be the intersection point of tangents on circle k in points B and D, respectively. If the lines AB and DF are parallel, prove that A, C and F are collinear.

2 Let n be a positive integer and let t be an integer. n distinct integers are written on a table. Bob, sitting in a room nearby, wants to know whether there exist some of these numbers such that their sum is equal to t. Alice is standing in front of the table and she wants to help him. At the beginning, she tells him only the initial sum of all numbers on the table. After that, in every move he says one of the 4 sentences: i. Is there a number on the table equal to k? ii. If a number k exists on the table, erase him. iii. If a number k does not exist on the table, add him. iv. Do the numbers written on the table can be arranged in two sets with equal sum of elements?
On these questions Alice answers yes or no, and the operations he says to her she does (if it is possible) and does not tell him did she do it. Prove that in less than $3n$ moves, Bob can find out whether there exist numbers initially written on the board such that their sum is equal to t

3 For an infinite sequence $a_1 < a_2 < a_3 < ...$ of positive integers we say that it is *nice* if for every positive integer n holds $a_{2n} = 2a_n$. Prove the following statements:

$a)$ If there is given a *nice* sequence and prime number $p > a_1$, there exist some term of the sequence which is divisible by p.

$b)$ For every prime number $p > 2$, there exist a *nice* sequence such that no terms of the sequence are divisible by p.

Day 2 May 15th

4 Determine the largest positive integer n which cannot be written as the sum of three numbers bigger than 1 which are pairwise coprime.

5 Let k be a circumcircle of triangle ABC $(AC < BC)$. Also, let CL be an angle bisector of angle ACB $(L \in AB)$, M be a midpoint of arc AB of circle k containing the point C, and let I be an incenter of a triangle ABC. Circle k cuts line MI at point K and circle with diameter CI at H. If the circumcircle of triangle CLK intersects AB again at T, prove that T, H and C are collinear.
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6 Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.