ICMC 2020-2021

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by mastermind.hk16

Round 1 20 December 2020

1 A set of points in the plane is called *sane* if no three points are collinear and the angle between any three distinct points is a rational number of degrees.

(a) Does there exist a countably infinite sane set $\mathcal{P}$?

(b) Does there exist an uncountably infinite sane set $\mathcal{Q}$?

Proposed by Tony Wang

2 Let A be a square matrix with entries in the field $\mathbb{Z}/p\mathbb{Z}$ such that $A^n - I$ is invertible for every positive integer n. Prove that there exists a positive integer m such that $A^m = 0$.

[i](A matrix having entries in the field $\mathbb{Z}/p\mathbb{Z}$ means that two matrices are considered the same if each pair of corresponding entries differ by a multiple of p.)[/i]

Proposed by Tony Wang

3 Let $s_n = \displaystyle\int_0^1 \sin^n(nx)\, dx$.

(a) Prove that $s_n \le \dfrac{2}{n}$ for all odd n.

(b) Find all the limit points of the sequence $s_1, s_2, s_3, \ldots$.

Proposed by Cristi Calin

4 Does there exist a set $\mathcal{R}$ of positive rational numbers such that every positive rational number is the sum of the elements of a unique finite subset of $\mathcal{R}$?

Proposed by Tony Wang

5 Find all composite positive integers m such that, whenever the product of two positive integers a and b is m, their sum is a power of 2.

Proposed by Harun Khan

6 There are $n+1$ squares in a row, labelled from 0 to n. Tony starts with k stones on square 0. On each move, he may choose a stone and advance the stone up to m squares where m is the number of stones on the same square (including itself) or behind it.

Tony's goal is to get all stones to square n. Show that Tony cannot achieve his goal in fewer than $\frac{n}{1}+\frac{n}{2}+\cdots+\frac{n}{k}$ moves.

Proposed by Tony Wang

Round 2 28 February 2021

1 Let S be a set with 10 distinct elements. A set T of subsets of S (possibly containing the empty set) is called *union-closed* if, for all $A, B \in T$, it is true that $A \cup B \in T$. Show that the number of union-closed sets T is less than 2^{1023}.

Proposed by Tony Wang

2 Let $p > 3$ be a prime number. A sequence of $p-1$ integers $a_1, a_2, \ldots, a_{p-1}$ is called *wonky* if they are distinct modulo p and $a_i a_{i+2} \not\equiv a_{i+1}^2 \pmod p$ for all $i \in \{1, 2, \ldots, p-1\}$, where $a_p = a_1$ and $a_{p+1} = a_2$. Does there always exist a wonky sequence such that

$$a_1a_2, \qquad a_1a_2 + a_2a_3, \qquad \ldots, \qquad a_1a_2 + \cdots + a_{p-1}a_1,$$

are all distinct modulo p?

Proposed by Harun Khan

3 Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions and X be a random variable such that $E(g(X)h(X)) = 0$ and $E(g(X)^2) \neq 0 \neq E(h(X)^2)$. Prove that

$$E(f(X)^2) \geq \frac{E(f(X)g(X))^2}{E(g(X)^2)} + \frac{E(f(X)h(X))^2}{E(h(X)^2)}.$$

You may assume that all expected values exist.

Proposed by Cristi Calin

4 Let $\mathbb{R}^2$ denote the Euclidean plane. A continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps circles to circles. (A point is not a circle.) Prove that it maps lines to lines.

Proposed by Tony Wang