Second Round Olympiad 2013

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– (A)

1 AB is a chord of circle ω, P is a point on minor arc AB, E, F are on segment AB such that $AE = EF = FB$. PE, PF meets ω at C, D respectively. Prove that $EF \cdot CD = AC \cdot BD$.

2 Let u, v be positive integers. Define sequence $\{a_n\}$ as follows: $a_1 = u + v$, and for integers $m \geq 1$,

$$\begin{cases} a_{2m} = a_m + u, \\ a_{2m+1} = a_m + v, \end{cases}$$

Let $S_m = a_1 + a_2 + \ldots + a_m (m = 1, 2, \ldots)$. Prove that there are infinitely many perfect squares in the sequence $\{S_n\}$.

3 n students take a test with m questions, where $m, n \geq 2$ are integers. The score given to every question is as such: for a certain question, if x students fails to answer it correctly, then those who answer it correctly scores x points, while those who answer it wrongly scores 0. The score of a student is the sum of his scores for the m questions. Arrange the scores in descending order $p_1 \geq p_2 \geq \ldots \geq p_n$. Find the maximum value of $p_1 + p_n$.

4 Let n, k be integers greater than 1, $n < 2^k$. Prove that there exist $2k$ integers none of which are divisible by n, such that no matter how they are separated into two groups there exist some numbers all from the same group whose sum is divisible by n.

– (B)

1 For any positive integer n , Prove that there is not exist three odd integer x, y, z satisfing the equation $(x + y)^n + (y + z)^n = (x + z)^n$.

– (C)

1 Let n be a positive odd integer , $a_1, a_2, \cdots, a_n$ be any permutation of the positive integers $1, 2, \cdots, n$. Prove that :$(a_1 - 1)(a_2^2 - 2)(a_3^3 - 3) \cdots (a_n^n - n)$ is an even number.

3 The integers $n > 1$ is given . The positive integer $a_1, a_2, \cdots, a_n$ satisfing condition :
(1) $a_1 < a_2 < \cdots < a_n$;

(2) $\frac{a_1^2+a_2^2}{2}, \frac{a_2^2+a_3^2}{2}, \cdots, \frac{a_{n-1}^2+a_n^2}{2}$ are all perfect squares .
Prove that :$a_n \geq 2n^2-1.$