Lusophon Mathematical Olympiad 2021

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–	Day 1
1	Juca has decided to call all positive integers with 8 digits as $sextalternados$ if it is a multiple of 30 and its consecutive digits have different parity. At the same time, Carlos decided to classify all $sextalternados$ that are multiples of 12 as $supersextalternados$. a) Show that $supersextalternados$ numbers don't exist. b) Find the smallest $sextalternado$ number.
2	Esmeralda has created a special knight to play on quadrilateral boards that are identical to chessboards. If a knight is in a square then it can move to another square by moving 1 square in one direction and 3 squares in a perpendicular direction (which is a diagonal of a 2×4 rectangle instead of 2×3 like in chess). In this movement, it doesn't land on the squares between the beginning square and the final square it lands on. A trip of the length n of the knight is a sequence of n squares $C1, C2, ..., Cn$ which are all distinct such that the knight starts at the $C1$ square and for each i from 1 to $n-1$ it can use the movement described before to go from the Ci square to the $C(i+1)$. Determine the greatest $N \in \mathbb{N}$ such that there exists a path of the knight with length N on a 5×5 board.
3	Let triangle ABC be an acute triangle with $AB \neq AC$. The bisector of BC intersects the lines AB and AC at points F and E, respectively. The circumcircle of triangle AEF has center P and intersects the circumcircle of triangle ABC at point D with D different to A. Prove that the line PD is tangent to the circumcircle of triangle ABC.
–	Day 2
4	Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^+$ such that $$x_1^2 - x_1x_2 + x_2^2 = x_2^2 - x_2x_3 + x_3^2 = x_3^2 - x_3x_4 + x_4^2 = x_4^2 - x_4x_5 + x_5^2 = x_5^2 - x_5x_1 + x_1^2$$ Prove that $x_1 = x_2 = x_3 = x_4 = x_5$.

5 There are 3 lines r, s and t on a plane. The lines r and s intersect perpendicularly at point A. the line t intersects the line r at point B and the line s at point C. There exist exactly 4 circumferences on the plane that are simultaneously tangent to all those 3 lines.

Prove that the radius of one of those circumferences is equal to the sum of the radius of the other three circumferences.

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6 A positive integer n is called $omopeiro$ if there exists n non-zero integers that are not necessarily distinct such that 2021 is the sum of the squares of those n integers. For example, the number 2 is not an $omopeiro$, because 2021 is not a sum of two non-zero squares, but 2021 is an $omopeiro$, because $2021 = 1^2 + 1^2 + \cdots + 1^2$, which is a sum of 2021 squares of the number 1.

Prove that there exist more than 1500 $omopeiro$ numbers.

Note: proving that there exist at least 500 $omopeiro$ numbers is worth 2 points.
