European Mathematical Cup 2021

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– Junior Division

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1 We say that a quadruple of nonnegative real numbers (a, b, c, d) is *balanced* if

$$a+b+c+d=a^2+b^2+c^2+d^2.$$

Find all positive real numbers x such that

$$(x-a)(x-b)(x-c)(x-d)\geq 0$$

for every balanced quadruple (a, b, c, d).

(Ivan Novak)

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2 Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Let X and Y be points on the minor arc BC of the circumcircle of ABC such that $|BX| = |XY| = |YC|$. Suppose that there exists a point N on the segment $\overline{AY}$ such that $|AB| = |AN| = |NC|$. Prove that the line NC passes through the midpoint of the segment $\overline{AX}$.

(Ivan Novak)

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3 Let ℓ be a positive integer. We say that a positive integer k is *nice* if $k! + \ell$ is a square of an integer. Prove that for every positive integer $n \geqslant \ell$, the set $\{1, 2, \dots, n^2\}$ contains at most $n^2 - n + \ell$ nice integers.

(Théo Lenoir)

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4 Let n be a positive integer. Morgane has coloured the integers $1, 2, \dots, n$. Each of them is coloured in exactly one colour. It turned out that for all positive integers a and b such that $a < b$ and $a + b \leqslant n$, at least two of the integers among a, b and $a + b$ are of the same colour. Prove that there exists a colour that has been used for at least $2n/5$ integers.

(Vincent Jugé)

– Senior Division

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1 Alice drew a regular 2021-gon in the plane. Bob then labeled each vertex of the 2021-gon with a real number, in such a way that the labels of consecutive vertices differ by at most 1. Then, for every pair of non-consecutive vertices whose labels differ by at most 1, Alice drew a diagonal connecting them. Let d be the number of diagonals Alice drew. Find the least possible value that d can obtain.

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2 Let ABC be a triangle and let D, E and F be the midpoints of sides BC, CA and AB, respectively.
Let $X \neq A$ be the intersection of AD with the circumcircle of ABC. Let Ω be the circle through D and X,
tangent to the circumcircle of ABC. Let Y and Z be the intersections of the tangent to Ω at D with the
perpendicular bisectors of segments DE and DF, respectively. Let P be the intersection of YE and ZF and
let G be the centroid of ABC. Show that the tangents at B and C to the circumcircle of ABC and the line PG are concurrent.

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3 Let $\mathbb{N}$ denote the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$x^2 - y^2 + 2y(f(x) + f(y))$$

is a square of an integer for all positive integers x and y.

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4 Find all positive integers d for which there exist polynomials $P(x)$ and $Q(x)$ with real coefficients such that degree of P equals d and

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2.$$
