China National Olympiad 22

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–	Day 1
1	Let a and b be two positive real numbers, and AB a segment of length a on a plane. Let C, D be two variable points on the plane such that $ABCD$ is a non-degenerate convex quadrilateral with $BC = CD = b$ and $DA = a$. It is easy to see that there is a circle tangent to all four sides of the quadrilateral $ABCD$. Find the precise locus of the point I.
2	Find the largest real number λ with the following property: for any positive real numbers p, q, r, s there exists a complex number $z = a + bi(a, b \in \mathbb{R})$ such that $$\lvert b\rvert \geq \lambda \lvert a\rvert \quad \text{and} \quad (pz^3 + 2qz^2 + 2rz + s) \cdot (qz^3 + 2pz^2 + 2sz + r) = 0.$$
3	Find all positive integers a such that there exists a set X of 6 integers satisfying the following conditions: for every $k = 1, 2, \ldots, 36$ there exist $x, y \in X$ such that $ax + y - k$ is divisible by 37.
–	Day 2
4	A conference is attended by $n(n \geq 3)$ scientists. Each scientist has some friends in this conference (friendship is mutual and no one is a friend of him/herself). Suppose that no matter how we partition the scientists into two nonempty groups, there always exist two scientists in the same group who are friends, and there always exist two scientists in different groups who are friends. A proposal is introduced on the first day of the conference. Each of the scientists' opinion on the proposal can be expressed as a non-negative integer. Everyday from the second day onwards, each scientists' opinion is changed to the integer part of the average of his/her friends' opinions from the previous day. Prove that after a period of time, all scientists have the same opinion on the proposal.
5	On a blank piece of paper, two points with distance 1 is given. Prove that one can use (only) straightedge and compass to construct on this paper a straight line, and two points on it whose distance is $\sqrt{2021}$ such that, in the process of constructing it, the total number of circles or straight lines drawn is at most 10. Remark: Explicit steps of the construction should be given. Label the circles and straight lines

in the order that they appear. Partial credit may be awarded depending on the total number of circles/lines.

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6 For integers $0 \leq a \leq n$, let $f(n,a)$ denote the number of coefficients in the expansion of $(x+1)^a(x+2)^{n-a}$ that is divisible by 3. For example, $(x+1)^3(x+2)^1 = x^4+5x^3+9x^2+7x+2$, so $f(4,3)=1$. For each positive integer n, let $F(n)$ be the minimum of $f(n,0), f(n,1), \dots, f(n,n)$.

(1) Prove that there exist infinitely many positive integer n such that $F(n) \geq \frac{n-1}{3}$.

(2) Prove that for any positive integer n, $F(n) \leq \frac{n-1}{3}$.
