Indonesia IMO TST

www.artofproblemsolving.com/community/c2742268
by parmenides51, somebodyyouusedtoknow, mathcyber124

–	Test 1

A Given a monic quadratic polynomial $Q(x)$, define

$$Q_n(x) = \underbrace{Q(Q(\cdots(Q(x))\cdots))}_{\text{compose } n \text{ times}}$$

for every natural number n. Let a_n be the minimum value of the polynomial $Q_n(x)$ for every natural number n. It is known that $a_n > 0$ for every natural number n and there exists some natural number k such that $a_k \neq a_{k+1}$.
(a) Prove that $a_n < a_{n+1}$ for every natural number n.
(b) Is it possible to satisfy $a_n < 2021$ for every natural number n?

Proposed by Fajar Yuliawan

C Five numbers are chosen from $\{1, 2, \ldots, n\}$. Determine the largest n such that we can always pick some of the 5 chosen numbers so that they can be made into two groups whose numbers have the same sum (a group may contain only one number).

G Given an acute triangle ABC. with H as its orthocenter, lines ℓ_1 and ℓ_2 go through H and are perpendicular to each other. Line ℓ_1 cuts BC and the extension of AB on D and Z respectively. Whereas line ℓ_2 cuts BC and the extension of AC on E and X respectively. If the line through D and parallel to AC and the line through E parallel to AB intersects at Y, prove that X, Y, Z are collinear.

N Prove that there exists a set $X \subseteq \mathbb{N}$ which contains exactly 2022 elements such that for every distinct $a, b, c \in X$ the following equality:

$$\gcd(a^n + b^n, c) = 1$$

is satisfied for every positive integer n.

–	Test 2

A Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$(a + b + c)(ab + bc + ca) + 3 \geq 4(a + b + c).$$

C Distinct pebbles are placed on a 1001×1001 board consisting of 1001^2 unit tiles, such that every unit tile consists of at most one pebble. The *pebble set* of a unit tile is the set of all pebbles situated in the same row or column with said unit tile. Determine the minimum amount of pebbles that must be placed on the board so that no two distinct tiles have the same *pebble set*.

It's already posted here (`https://artofproblemsolving.com/community/c6h2742895_simple_inequality`).

G Given that ABC is a triangle, points $A_i, B_i, C_i \ (i \in \{1,2,3\})$ and O_A, O_B, O_C satisfy the following criteria:

a) $ABB_1A_2, BCC_1B_2, CAA_1C_2$ are rectangles not containing any interior points of the triangle ABC,
b) $\frac{AB}{BB_1} = \frac{BC}{CC_1} = \frac{CA}{AA_1}$,
c) $AA_1A_3A_2, BB_1B_3B_2, CC_1C_3C_2$ are parallelograms, and
d) O_A is the centroid of rectangle BCC_1B_2, O_B is the centroid of rectangle CAA_1C_2, and O_C is the centroid of rectangle ABB_1A_2.

Prove that $A_3O_A, B_3O_B,$ and C_3O_C concur at a point.

Proposed by Farras Mohammad Hibban Faddila

N Let n be a natural number, with the prime factorisation

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

where $p_1, \ldots, p_r$ are distinct primes, and e_i is a natural number. Define

$$rad(n) = p_1 p_2 \cdots p_r$$

to be the product of all distinct prime factors of n. Determine all polynomials $P(x)$ with rational coefficients such that there exists infinitely many naturals n satisfying $P(n) = rad(n)$.

– Test 3

A Let a and b be two positive reals such that the following inequality

$$ax^3 + by^2 \geq xy - 1$$

is satisfied for any positive reals $x, y \geq 1$. Determine the smallest possible value of $a^2 + b$.

Proposed by Fajar Yuliawan

C A $3 \times 3 \times 4$ cuboid is constructed out of 36 white-coloured unit cubes. Then, all six of the cuboid's sides are coloured red. After that, the cuboid is dismantled into its constituent unit cubes. Then, randomly, all said unit cubes are constructed into the cuboid of its original size (and position).

a) How many ways are there to position eight of its corner cubes so that the apparent sides of eight corner cubes are still red? (Cube rotations are still considered distinct configurations, and the position of the cuboid remains unchanged.)

b) Determine the probability that after the reconstruction, all of its apparent sides are still red-coloured. (The cuboid is still upright, with the same dimensions as the original cuboid, without rotation.)

The problem might have multiple interpretations. We agreed that this problem's wording was a bit ambiguous. Here's the original Indonesian version:
Suatu balok berukuran $3 \times 3 \times 4$ tersusun dari 36 kubus satuan berwarna putih. Kemudian keenam permukaan balok diwarnai merah. Setelah itu, balok yang tersusun dari kubus-kubus satuan tersebut dibongkar. Kemudian, secara acak, semua kubus satuan disusun lagi menjadi balok seperti balok semula.

a) Ada berapa cara menempatkan kedelapan kubus satuan yang berasal dari pojok sehingga kedelapan kubus di pojok yang tampak tetap berwarna merah? (Rotasi kubus dianggap konfigurasi yang berbeda, namun posisi balok tidak diubah.)

b) Tentukan probabilitas balok yang tersusun lagi semua permukaannya berwarna merah. (Balok tegak tetap tegak dan balok tetap dalam suatu posisi.)

G Let AB be the diameter of circle Γ centred at O. Point C lies on ray $\overrightarrow{AB}$. The line through C cuts circle Γ at D and E, with point D being closer to C than E is. OF is the diameter of the circumcircle of triangle BOD. Next, construct CF, cutting the circumcircle of triangle BOD at G. Prove that O, A, E, G are concyclic.

(Possibly proposed by Pak Wono)

N Given positive odd integers m and n where the set of all prime factors of m is the same as the set of all prime factors n, and $n|m$. Let a be an arbitrary integer which is relatively prime to m and n. Prove that:

$$o_m(a) = o_n(a) \times \frac{m}{\gcd(m, a^{o_n(a)} - 1)}$$

where $o_k(a)$ denotes the smallest positive integer such that $a^{o_k(a)} \equiv 1 \pmod{k}$ holds for some natural number $k > 1$.

– Test 4

A Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(a^2) - f(b^2) \leq (f(a) + b)(a - f(b))$$

for all $a, b \in \mathbb{R}$.

C Let A be a subset of $\{1, 2, \ldots, 2020\}$ such that the difference of any two distinct elements in A is not prime. Determine the maximum number of elements in set A.

G In a nonisosceles triangle ABC, point I is its incentre and Γ is its circumcircle. Points E and D lie on Γ and the circumcircle of triangle BIC respectively such that AE and ID are both perpendicular to BC. Let M be the midpoint of BC, N be the midpoint of arc BC on Γ containing A, F is the point of tangency of the $A-$excircle on BC, and G is the intersection of line DE with Γ. Prove that lines GM and NF intersect at a point located on Γ.

(Possibly proposed by Farras Faddila)

N For each natural number n, let $f(n)$ denote the number of ordered integer pairs (x, y) satisfying the following equation:

$$x^2 - xy + y^2 = n.$$

a) Determine $f(2022)$.
b) Determine the largest natural number m such that m divides $f(n)$ for every natural number n.
